Regularity results for the Primitive Equations of the ocean

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November 3, 2006

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Abstract. We consider the linear Primitive Equations of the ocean in the three dimensional space, with horizontal periodic and vertical Dirichlet boundary conditions. Thanks to Fourier transforms we are able to calculate explicitly the pressure term. We then state existence, unicity and regularity results for the linear time-depending Primitive Equations, with low-regularity right-hand side.

1 Introduction and main results

We establish regularity results for the linear Primitive Equations (PE) of the ocean in the three dimensional space. For the nonlinear PE, the first work of Lions, Temam and Wang [4] and the further paper by Temam and Ziane [7] proved global existence of weak solutions and local existence and unicity of strong solutions. The regularity of the linear Stokes-type problem related to the PE has been studied by Ziane [8], Hu, Temam and Ziane [3] and Temam and Ziane [7]. For diverse boundary conditions, the authors prove the regularity of weak solutions, when the right-hand side stays in L^2.
Our work is motivated by the following remark: many problems involving the PE of the ocean (such as numerical ocean modelling, assimilation of surface data or more theoretically controllability of the PE) need the calculation, or

at least the estimation, of the pressure term. But in the previous studies of the PE, the regularity of the pressure is not explicitly investigated. Thus the aim of this paper is to calculate explicitly the most singular part of the pressure term in order to obtain more precise regularity results, in particular with less regular right-hand side. The analogous of this question for the full Stokes problem has been addressed by Fabre and Lebeau [1].
The paper is organized as follows: in the rest of this section our main results are stated, in section 2 we present some preliminaries, in section 3 we prove theorem 1, some further remarks and results are provided in section 4.

1.1 The Primitive Equations of the ocean

The Primitive Equations of the ocean are at the base of general ocean circulation models, intensively used by oceanographers. The linear equations we are interested with are the following:

$$\begin{cases} \partial_t u - \nu\Delta u - \alpha\, v + \partial_x p = f_1 & \text{in } \Omega\times(0,T) \\ \partial_t v - \nu\Delta v + \alpha\, u + \partial_y p = f_2 & \\ \partial_z p - \beta\,\theta = 0 & \\ \partial_t \theta - \nu\Delta\theta + \gamma\, w = f_3 & \\ w(x,y,z,t) = -\int_0^z \partial_x u(x,y,z',t) + \partial_y v(x,y,z',t)\, dz' & \end{cases} \tag{1}$$

with the following initial conditions:

$$U(t=0) = U_0, \qquad \theta(t=0) = \theta_0 \text{ in } \Omega \tag{2}$$

and boundary conditions:

$$\begin{cases} u, v, w, \theta, p \text{ are periodic in } x, y \\ u = 0, v = 0, \theta = 0 \quad \text{on } \mathbb{T}^2 \times \{z = 0, z = a\} \times (0,T) \\ \int_{z=0}^a \partial_x u + \partial_y v\, dz = 0 \quad \text{on } \mathbb{T}^2 \times (0,T) \end{cases} \tag{3}$$

where
– Ω is a horizontally periodic and vertically bounded ocean basin: $\Omega = \mathbb{T}^2 \times (0,a)$, with $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ the bidimensional torus;
– $U = (u,v)$ is the horizontal velocity of the fluid, w its vertical velocity;
– θ is the temperature around a vertical temperature profile $\theta = \widetilde{\theta} - \theta_b - z\frac{\theta_b - \theta_a}{a}$, with $\widetilde{\theta}$ the real temperature, θ_a and θ_b top and bottom boundary conditions for $\widetilde{\theta}$;
– p is the pressure;
– $F = (f_1, f_2, f_3)$ is a given forcing term;

– α is the constant Coriolis parameter;
– ν is the kinematic viscosity and the temperature diffusion parameter;
– β is a physical constant, depending on the gravity constant;
– γ is a constant.

Remark 1 *1. We have assumed, without loss of generality, that the salinity does not appear in the state equation, so that it is a passive tracer (see [7] for the full equations).*

2. In order to lighten notations, we have assumed that the kinematic horizontal and vertical viscosity and the diffusion parameter in the temperature equations are equal, we thus have similar results with different values of these parameters.

3. We choose to use Dirichlet boundary conditions because these are realistic physical conditions; moreover this enables us to consider low regularity forcings F.

We will also use the stationary linear model $\mathcal{S}_\lambda$ for spectral study, with $\lambda \in \mathbb{C}$:

$$\mathcal{S}_\lambda(u, v, \theta) = F$$
$$\Updownarrow$$
$$\begin{cases} \lambda u - \nu\Delta u - \alpha\, v + \partial_x p = f_1 & \text{in } \Omega \\ \lambda v - \nu\Delta v + \alpha\, u + \partial_y p = f_2 & \\ \partial_z p - \beta\,\theta = 0 & \\ \lambda\theta - \nu\Delta\theta + \gamma\, w = f_3 & \\ w(x, y, z) = -\int_0^z \partial_x u(x, y, z') + \partial_y v(x, y, z')\, dz' & \end{cases} \tag{4}$$

with stationary boundary conditions:

$$\begin{cases} u, v, w, \theta, p \text{ are periodic in } x, y \\ u = 0, v = 0, \theta = 0 \quad \text{on } \mathbb{T}^2 \times \{z = 0, z = a\} \\ \int_{z=0}^a \partial_x u + \partial_y v\, dz = 0 \quad \text{on } \mathbb{T}^2 \end{cases} \tag{5}$$

1.2 Some functional spaces

Let us now introduce some functional spaces

Definition 1 *Forall $s \in \mathbb{R}$*

$$\mathcal{H}^s(\Omega) = \Big\{ \begin{array}{l} f(x, y, z) = \sum_{k\in\mathbb{N}^*, \zeta\in\mathbb{Z}^2} f_{k,\zeta}\, e_k(z)\, e_\zeta(x, y), (x, y, z) \in \Omega, \\ \sum_{k\in\mathbb{N}^*, \zeta\in\mathbb{Z}^2} (1 + \nu k^2 + \nu|\zeta|^2)^s |f_{k,\zeta}|^2 < \infty \Big\} \end{array} \tag{6}$$

with:

$$\begin{array}{rcl} e_k(z) & = & \sqrt{\frac{2}{a}}\sin(\frac{k\pi z}{a}) \quad \forall z\in(0,a) \\ e_\zeta(x,y) & = & \frac{1}{2\pi}e^{i(\xi x+\eta y)} \qquad \forall (x,y)\in\mathbb{T}^2 \end{array} \tag{7}$$

forall $k \in \mathbb{N}^*$ *and* $\zeta = (\xi, \eta) \in \mathbb{Z}^2$.

For $f \in \mathcal{H}^s$ we denote

$$\|f\|_s^2 = \sum_{k\in\mathbb{N}^*,\zeta\in\mathbb{Z}^2} (1+\nu k^2+\nu|\zeta|^2)^s |f_{k,\zeta}|^2 \tag{8}$$

$\mathcal{H}^s(\Omega)$ is a Hilbert space with the following inner product:

$$\langle f, g\rangle_s = \sum_{k\in\mathbb{N}^*,\zeta\in\mathbb{Z}^2} (1+\nu k^2+\nu|\zeta|^2)^s f_{k,\zeta}\, \overline{g_{k,\zeta}}. \tag{9}$$

The following characterization holds true, where $H^s(\Omega)$ denotes the usual Sobolev space:

Lemma 1

$$\begin{array}{rcl} -\frac{3}{2} < s < \frac{1}{2} \;\Rightarrow\; \mathcal{H}^s(\Omega) & = & H^s(\Omega) \\ \frac{1}{2} < s < \frac{5}{2} \;\Rightarrow\; \mathcal{H}^s(\Omega) & = & \{f\in H^s(\Omega), f|_{z=0} = f|_{z=a} = 0\} \end{array} \tag{10}$$

We define also the following classical spaces (see [2], [6] or [3]):

Definition 2 *Let*

$$\begin{array}{rcl} E_1 & = & \{U=(u,v)\in\mathcal{C}^\infty(\Omega)^2, u,v \textit{ periodic in } x,y, \\ & & \quad u=0, v=0 \textit{ on } \mathbb{T}^2\times\{z=0,z=a\} \\ & & \quad \int_0^a \partial_x u(x,y,z') + \partial_y v(x,y,z')\,dz' = 0, \forall (x,y)\in\mathbb{T}^2\} \\ E_2 & = & \{\theta\in\mathcal{C}^\infty(\Omega), \theta \textit{ periodic in } x,y, \\ & & \quad \theta = 0 \textit{ on } \mathbb{T}^2\times\{z=0,z=a\}\} \end{array} \tag{11}$$

Then $\mathcal{H}_1$ (respectively $\mathcal{H}_2$) is defined to be the closure of E_1 in $L^2(\Omega)^2$ (resp. $L^2(\Omega)$), and $\mathcal{V}_1$ (resp. $\mathcal{V}_2$) is the closure of E_1 (resp. E_2) in $H^1(\Omega)^2$ (resp. $H^1(\Omega)$), and finally $\mathcal{H} = \mathcal{H}_1\times\mathcal{H}_2$, $\mathcal{V} = \mathcal{V}_1\times\mathcal{V}_2$.

Inner products on $\mathcal{H}$ and $\mathcal{V}$ are:

$$\begin{array}{rcl} (X,X')_{\mathcal{H}} & = & (u,u')_{L^2(\Omega)} + (v,v')_{L^2(\Omega)} + \frac{\beta}{\gamma}(\theta,\theta')_{L^2(\Omega)} \\ & = & \int_\Omega (u\overline{u'} + v\overline{v'} + \frac{\beta}{\gamma}\theta\overline{\theta'})\,dx\,dy\,dz \\ (X,X')_{\mathcal{V}} & = & (u,u')_{H_0^1(\Omega)} + (v,v')_{H_0^1(\Omega)} + \frac{\beta}{\gamma}(\theta,\theta')_{H_0^1(\Omega)} \\ & = & \int_\Omega (\nabla u.\nabla\overline{u'} + \nabla v.\nabla\overline{v'} + \frac{\beta}{\gamma}\nabla\theta.\nabla\overline{\theta'})\,dx\,dy\,dz \end{array} \tag{12}$$

1.3 Results

Our main result states as follows:

Theorem 1 *Let* $\sigma \in]-\frac{3}{2}, \frac{1}{2}[$, $\sigma \neq -\frac{1}{2}$.
Let $F(t) = (f_1, f_2, f_3) \in (L^2(\mathbb{R}; \mathcal{H}^\sigma)^3)$ *with Support* $(F) \subset \{t \geq 0\}$.
There exists a unique

$$X(t) = (u, v, \theta) \in (L^2(\mathbb{R}; \mathcal{H}^{\sigma+2})^3), \quad \textit{Support } (X) \subset \{t \geq 0\} \tag{13}$$

and there exists a unique (up to a distribution depending only on t*) pressure*

$$p(t) \in \mathcal{D}'(\mathbb{R} \times \Omega), \quad \textit{Support } (p) \subset \{t \geq 0\} \tag{14}$$

so that the following equation holds true, in the sense of distributions in $\mathbb{R} \times \Omega$

$$\begin{cases} \partial_t u - \nu \Delta u - \alpha v + \partial_x p = f_1 \\ \partial_t v - \nu \Delta v + \alpha u + \partial_y p = f_2 \\ \partial_z p - \beta \theta = 0 \\ \partial_t \theta - \nu \Delta \theta + \gamma w = f_3 \end{cases} \tag{15}$$

$$\textit{with } w(z) = -\int_0^z (\partial_x u + \partial_y v) \textit{ and } w(a) = 0$$

Moreover

$$\|X\|_{(L^2(\mathbb{R}; \mathcal{H}^{\sigma+2}))^3} \leq C \|F\|_{(L^2(\mathbb{R}; \mathcal{H}^\sigma))^3} \tag{16}$$

and the temperature θ *verifies*

$$\partial_t \theta \in L^2(\mathbb{R}; \mathcal{H}^\sigma) \quad \textit{and} \quad \|\partial_t \theta\|_{L^2(\mathbb{R}; \mathcal{H}^\sigma)} \leq C \|F\|_{(L^2(\mathbb{R}; \mathcal{H}^\sigma))^3} \tag{17}$$

The pressure p *verifies*

$$p(t, x, y, z) = c(t) + q(t, x, y) + \beta \int_0^z \theta(t, x, y, z')\, dz' \tag{18}$$

with

$$c(t) \in \mathcal{D}'(\mathbb{R}), \quad \textit{Support } (c) \subset \{t \geq 0\} \tag{19}$$

and

- *for* $\sigma \in]-\frac{1}{2}, \frac{1}{2}[$ *we have*

$$q(t, x, y) \in L^2(\mathbb{R}; H^{\sigma+1}(\mathbb{T}^2)) \quad \textit{and} \quad \|q\|_{L^2(\mathbb{R}; H^{\sigma+1}(\mathbb{T}^2))} \leq C \|F\|_{(L^2(\mathbb{R}; \mathcal{H}^\sigma))^3} \tag{20}$$

- *for* $\sigma \in]-\frac{3}{2}, -\frac{1}{2}[$ *we have*

$$q(t, x, y) = q_1(t, x, y) + q_2(t, x, y)$$

$$q_2(t, x, y) \in L^2(\mathbb{R}; H^{\sigma+1}(\mathbb{T}^2)) \textit{ and } \|q_2\|_{L^2(\mathbb{R}; H^{\sigma+1}(\mathbb{T}^2))} \leq C \|F\|_{(L^2(\mathbb{R}; \mathcal{H}^\sigma))^3}$$
$$q_1(t, x, y) \in H^{\sigma/2+1/4}(\mathbb{R}; H^1(\mathbb{T}^2)) \textit{ and } \|q_1\|_{H^{\sigma/2+1/4}(\mathbb{R}; H^1(\mathbb{T}^2))} \leq C \|F\|_{(L^2(\mathbb{R}; \mathcal{H}^\sigma))^3} \tag{21}$$

Remark 2 *1. The regularity exponent σ.*

- With a forcing term $F \in (L^2(\mathbb{R};\mathcal{H}^\sigma)^3)$, we cannot have more than $X \in (L^2(\mathbb{R};\mathcal{H}^{\sigma+2})^3)$. Thus the boundary condition $X|_{z=0,z=a} = 0$ is well defined only if $\sigma + 2 > \frac{1}{2}$, i.e. $\sigma > -\frac{3}{2}$.
- $\sigma = -\frac{1}{2}$ is a critical exponent for the regularity of the pressure, whose description is more technical.
- Considering only $\sigma < \frac{1}{2}$ enables us to use the spaces $\mathcal{H}^\sigma$ and to do explicit calculations.

2. *An explicit formula for q_1.*

   Actually, for $\sigma \in]-\frac{3}{2}, -\frac{1}{2}[$, we will prove a more precise result than (21), namely

$$q(t,x,y) = q_1(t,x,y) + q_2(t,x,y), \quad q_2(t,x,y) \in L^2(\mathbb{R}; H^{\sigma+1}(\mathbb{T}^2)) \tag{22}$$

   where q_1 is explicit as a function of F (see remark 4.1 and formula (174)).

3. *Formula (18).*

   In formula (18) for p, q is the value of p at $z = 0$. We can replace q either by $p(t,x,y,z_0)$, for any $z_0 \in [0,a]$, or by $\int_0^a p(t,x,y,z)\,dz$, the results remain the same.

4. *Maximal estimates.*

   For $\sigma > -\frac{1}{2}$, we have $\partial_x p, \partial_y p \in L^2(\mathbb{R};\mathcal{H}^\sigma)$, so that the pressure gradient term can be seen as a forcing term and we have the following maximal estimates

$$\|X\|_{(L^2(\mathbb{R};\mathcal{H}^{\sigma+2}))^3} + \|\partial_t X\|_{(L^2(\mathbb{R};\mathcal{H}^{\sigma}))^3} \le C\,\|F\|_{(L^2(\mathbb{R};\mathcal{H}^{\sigma}))^3} \tag{23}$$

   However, for $\sigma \in]-\frac{3}{2}, -\frac{1}{2}[$, the maximal estimate is wrong (see remark 4.3).

We prove also the following corollary for the Cauchy problem with $\sigma = -1$:

Corollary 1 *Let $\varphi(t) \in \mathcal{C}_c^\infty(]0,T[)$, $F(t) = (f_1, f_2, f_3) \in (L^2(0,T;\mathcal{H}^{-1}))^3$, $X_0 \in \mathcal{H}$. Let (X,p) be the unique solution of equation (15) with*

$$\begin{array}{l} X = (u,v,\theta) \in L^2(0,T;\mathcal{V}) \cap \mathcal{C}([0,T];\mathcal{H}), \quad X(t=0) = X_0 \\ p \in \mathcal{D}'(0,T;L^2(\Omega)) \end{array} \tag{24}$$

Then φp is rewritten as

$$\varphi p(t,x,y,z) = c(t) + q(t,x,y) + \beta \int_0^z \theta(t,x,y,z')\,dz', \text{ with } c(t) \in \mathcal{D}'(\mathbb{R}) \tag{25}$$

with $q(t) \in H^{-1/4}(0,T;H^1(\mathbb{T}^2))$ *and we have*

$$\begin{array}{c} q(t) \in L^2(0,T;L^2(\mathbb{T}^2)) \\ \Updownarrow \\ \Delta_2^{-1}\big[\int_0^a (\partial_t - \nu\Delta)^{-1}[\varphi\partial_x f_1 + \varphi\partial_y f_2]\,dz\big] \in L^2(0,T;L^2(\mathbb{T}^2)) \end{array} \tag{26}$$

where Δ_2 *is the horizontal Laplacian operator, defined by* $\Delta_2\psi = \partial_{xx}\psi + \partial_{yy}\psi$.

Remark 3 *The preceding Cauchy problem can easily be addressed thanks to classical variational methods, but it gives less precise results regarding the pressure, see lemma 5 and remark 5.*

2 Preliminary results

2.1 The Primitive Equations operator

Multiplying equation (4) by $\gamma\overline{u'}$, $\gamma\overline{v'}$, $\gamma\overline{w'}$, $\beta\overline{\theta'}$ (with $X' = (u', v', \theta') \in \mathcal{V}$) and integrating by parts (using boundary conditions (3)), we obtain formally:

$$\begin{array}{c} \mathcal{S}_\lambda(X) = F \\ \Updownarrow \\ \lambda(X,X')_{\mathcal{H}} + \nu(X,X')_{\mathcal{V}} + \beta B(X,X') + \alpha C(X,X') = (F,X')_{\mathcal{H}}, \quad \forall X' \in \mathcal{V} \end{array} \tag{27}$$

where B and C are given by:

$$\begin{array}{l} B(X,X') = -(\theta,w')_{L^2(\Omega)} + (w,\theta')_{L^2(\Omega)} \\ \qquad \text{with } w = -\int_0^z \partial_x u + \partial_y v, \quad w' = -\int_0^z \partial_x u' + \partial_y v' \\ C(X,X') = -(v,u')_{L^2(\Omega)} + (u,v')_{L^2(\Omega)} \end{array} \tag{28}$$

and

$$\begin{array}{llll} B(X,X) & = \int_\Omega(-\theta\overline{w} + w\overline{\theta}) & = 2i\Im(\int_\Omega w\overline{\theta}) & \in i\mathbb{R} \\ C(X,X) & = \int_\Omega(-v\overline{u} + u\overline{v}) & = 2i\Im(\int_\Omega u\overline{v}) & \in i\mathbb{R} \end{array} \tag{29}$$

We define then $A(X,X') = (X,X')_{\mathcal{V}}$. The operator $P = \nu A + \beta B + \alpha C$, called Primitive Equations operator, maps $\mathcal{V}$ to $\mathcal{V}'$, and for all $(X,X') \in \mathcal{V}$ we have

$$\begin{aligned} \langle P(X), X'\rangle_{\mathcal{V}',\mathcal{V}} &= \nu(X,X')_{\mathcal{V}} + \beta B(X,X') + \alpha C(X,X') \\ \langle (\lambda + P)(X), X'\rangle_{\mathcal{V}',\mathcal{V}} &= \lambda(X,X')_{\mathcal{H}} + \nu(X,X')_{\mathcal{V}} + \beta B(X,X') + \alpha C(X,X') \end{aligned} \tag{30}$$

Remark 4 *Operator A corresponds to the uncoupled Stokes-type equation obtained from (4) with $\alpha = \beta = \gamma = 0$, B corresponds to the coupling (via the parameters β and γ) between the vertical velocity w and the temperature θ and C is the Coriolis operator.*

We have:

Lemma 2 *The mapping*

$$\Phi : (X, X') \mapsto \langle P(X), X' \rangle_{\mathcal{V}',\mathcal{V}}$$

is continuous on $\mathcal{V}^2$. More precisely

$$|\langle P(X), X' \rangle_{\mathcal{V}',\mathcal{V}}| \leq (\nu + 2\frac{a^2}{\pi}\sqrt{\beta\gamma} + \frac{2\alpha a^2}{\pi^2})\|X\|_{\mathcal{V}}\|X'\|_{\mathcal{V}} \tag{31}$$

Proof. Let $X = (u, v, \theta)$ and $X' = (u', v', \theta')$ be in $\mathcal{V}$. We have clearly:

$$\begin{array}{rcl} \langle A(X), X' \rangle_{\mathcal{V}',\mathcal{V}} & = & (X, X')_{\mathcal{V}} \leq \|X\|_{\mathcal{V}}\|X'\|_{\mathcal{V}} \\ |B(X, X')| & \leq & \|\theta\|_{L^2(\Omega)}\|w'\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}\|\theta'\|_{L^2(\Omega)} \\ |C(X, X')| & \leq & \|v\|_{L^2(\Omega)}\|u'\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\|v'\|_{L^2(\Omega)} \end{array} \tag{32}$$

For all $\varphi \in H_0^1(\Omega)$ the following Poincaré inequality holds:

$$\|\varphi\|^2_{L^2(\Omega)} \leq \frac{a^2}{\pi^2}\|\nabla\varphi\|^2_{L^2(\Omega)} \tag{33}$$

Thus we obtain

$$\|u\|_{L^2(\Omega)} \leq \|X\|_{\mathcal{H}} \leq \frac{a}{\pi}\|X\|_{\mathcal{V}}, \qquad \|\theta\|_{L^2(\Omega)} \leq \sqrt{\frac{\gamma}{\beta}}\|X\|_{\mathcal{H}} \leq \frac{a}{\pi}\sqrt{\frac{\gamma}{\beta}}\|X\|_{\mathcal{V}} \tag{34}$$

Using Cauchy-Schwarz inequality, we obtain

$$\begin{array}{rcl} \|w\|^2_{L^2(\Omega)} & = & \| \int_0^z \partial_x u + \partial_y v\|^2_{L^2(\Omega)} \\ & \leq & a^2\|X\|^2_{\mathcal{V}} \end{array} \tag{35}$$

A straightforward calculation gives the desired conclusion.
□

2.2 Qualitative spectral study

Definition 3 *We call eigenvalue of $-P : \mathcal{V} \to \mathcal{V}'$ a complex number λ so that $\lambda + P$ is not injective. We denote $\mathbb{V}_P$ the set of the eigenvalues of $-P$:*

$$\mathbb{V}_P = \{\lambda \in \mathbb{C},\, \exists X \in \mathcal{V},\, X \neq 0,\, \langle \lambda X + P(X), X' \rangle_{\mathcal{V}',\mathcal{V}} = 0\, \forall X' \in \mathcal{V}\} \quad (36)$$

Lemma 3 *We have the following inclusion:*

$$\mathbb{V}_P \subset \left\{ \lambda \in \mathbb{C}, \Re(\lambda) \leq -\frac{\nu\pi^2}{a^2} \text{ and } |\Im(\lambda)| \leq 2\alpha + 2a\sqrt{\beta\gamma}\sqrt{-\frac{\Re(\lambda)}{\nu}} \right\} \quad (37)$$

Proof. If λ is an eigenvalue of $-P$, then there exists $X \in \mathcal{V}, X \neq 0$ such that

$$\lambda(X, X')_{\mathcal{H}} + \nu(X, X')_{\mathcal{V}} + \beta B(X, X') + \alpha C(X, X') = 0, \forall X' \in \mathcal{V} \quad (38)$$

Using (29) and (38) with $X' = X$, we have

$$\begin{array}{l} \Re(\lambda)\|X\|^2_{\mathcal{H}} + \nu\|X\|^2_{\mathcal{V}} = 0 \\ \Im(\lambda)\|X\|^2_{\mathcal{H}} + 2\beta\Im(\int_\Omega w\overline{\theta}) + 2\alpha\Im(\int_\Omega u\overline{v}) = 0 \end{array} \quad (39)$$

Thanks to (34) $\|X\|^2_{\mathcal{H}} \leq \frac{a^2}{\pi^2}\|X\|^2_{\mathcal{V}}$, we obtain:

$$\Re(\lambda) = -\frac{\nu\|X\|^2_{\mathcal{V}}}{\|X\|^2_{\mathcal{H}}} \leq -\frac{\nu\pi^2}{a^2} \quad (40)$$

With (34) and (35) we have:

$$\begin{array}{rcl} |\Im(\lambda)| & = & 2\beta|\Im(\int_\Omega w\overline{\theta})|/\|X\|^2_{\mathcal{H}} + 2\alpha|\Im(\int_\Omega u\overline{v})|/\|X\|^2_{\mathcal{H}} \\ & \leq & 2\beta\|w\|_{L^2(\Omega)}\|\theta\|_{L^2(\Omega)}/\|X\|^2_{\mathcal{H}} + 2\alpha\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}/\|X\|^2_{\mathcal{H}} \\ & \leq & 2a\sqrt{\beta\gamma}\|X\|_{\mathcal{V}}/\|X\|_{\mathcal{H}} + 2\alpha \\ & = & 2\alpha + 2a\sqrt{\beta\gamma}\sqrt{-\frac{\Re(\lambda)}{\nu}} \end{array} \quad (41)$$

From (40) and (41) we get that (37) holds true.
□

2.3 First existence and unicity results

Let us finish this section by stating two lemmas, whose proofs are based on very classical use of the variational method, as in [4] or in [7].

Lemma 4 *If $\lambda \in \mathbb{C} \setminus \mathbb{V}_P$ and $Y = (y_1, y_2, y_3) \in (H^{-1}(\Omega))^3$ then there exists a unique $X = (u, v, \theta) \in \mathcal{V}$ and there exists a pressure $p(x, y, z) \in L^2(\Omega)$, unique up to a constant, so that*

$$\begin{cases} \lambda u - \nu\Delta u - \alpha v + \partial_x p = y_1 & \text{in } \Omega \\ \lambda v - \nu\Delta v + \alpha u + \partial_y p = y_2 \\ \partial_z p - \beta\theta = 0 \\ \lambda\theta - \nu\Delta\theta + \gamma w = y_3 \\ w(x, y, z) = -\int_0^z \partial_x u(x, y, z') + \partial_y v(x, y, z')\, dz' \end{cases} \tag{42}$$

Lemma 5 *Let $T > 0$, $X_0 \in \mathcal{H}$ and $F = (f_1, f_2, f_3) \in L^2(0, T, (H^{-1}(\Omega))^3)$. Then there exists a unique*

$$X = (u, v, \theta) \in L^2(0, T; \mathcal{V}) \cap \mathcal{C}([0, T]; \mathcal{H}) \tag{43}$$

and there exists a pressure

$$p \in \mathcal{D}'(0, T; L^2(\Omega)) \tag{44}$$

unique (up to a time distribution), such that the following equation holds true in the sense of distributions in $\Omega \times (0, T)$:

$$\begin{cases} \partial_t u - \nu\Delta u - \alpha v + \partial_x p = f_1 \\ \partial_t v - \nu\Delta v + \alpha u + \partial_y p = f_2 \\ \partial_z p - \beta\theta = 0 \\ \partial_t \theta - \nu\Delta\theta + \gamma w = f_3 \\ \text{with } w(z) = -\int_0^z \partial_x u(z') + \partial_y v(z')\, dz' \end{cases} \tag{45}$$

and

$$(u, v, \theta)|_{t=0} = X_0 \tag{46}$$

Remark 5 *The derivative $\frac{dX}{dt}$ is in $H^{-1}(0, T, (H_0^1(\Omega))^3)$ and equation (45) tells us in particular:*

$$\nabla p \in L^2(0, T, (H^{-1}(\Omega))^3) + H^{-1}(0, T, (H_0^1(\Omega))^3) \tag{47}$$

3 Proof of theorem 1

The proof is organized as follows. In section 3.1 we take the Fourier-Laplace transform of the equation, first in the horizontal coordinates, then in the vertical one and finally in time. Spectral parameters are then introduced, $\lambda = i\tau$ and $\omega = \lambda + \nu\zeta^2$, where τ is the Laplace parameter and ζ the horizontal Fourier variable. In section 3.2 we introduce the function $M_\sigma(\lambda,\zeta)$ and we prove preliminary estimates for this function. In section 3.3 we study the uncoupled system, ie (69) ie with $\alpha=\beta=\gamma=0$. This is the core of the proof. We will use the function M_σ in order to establish in theorem 2 optimal estimates for the uncoupled system depending on the values and asymptotics of the parameters and the vertical Fourier variable. In section 3.4 we use the results of section 3.3 to establish estimates for the coupled system. We conclude the proof in section 3.5.

3.1 First reductions

Estimation (17) for temperature is straightforward: if (16) holds, then $w\in L^2(\mathbb{R};\mathcal{H}^\sigma)$ and

$$\|w\|_{L^2(\mathbb{R};\mathcal{H}^\sigma)} \;\le\; C\,\|F\|_{L^2(\mathbb{R};\mathcal{H}^\sigma)} \tag{48}$$

then the temperature satisfies

$$\partial_t\theta-\nu\Delta\theta=f_3-\gamma w \tag{49}$$

and (17)follows easily. Thus it is sufficient to prove existence, unicity, (18), (16), (20) and (21).

Fourier transform in space. For $f\in\mathcal{D}'(\mathbb{R}\times\Omega)$, we write

$$f(t,x,y,z)=\sum_{\zeta\in\mathbb{Z}^2} f_\zeta(t,z)\,e^{i\zeta\cdot(x,y)} \tag{50}$$

Equation (15) is equivalent to the following equations, with parameter $\zeta=(\xi,\eta)\in\mathbb{Z}^2$:

$$\begin{cases}\partial_t u_\zeta-\nu\partial_{zz}u_\zeta+\nu\zeta^2u_\zeta-\alpha v_\zeta+i\xi p_\zeta=f_{1,\zeta}\\ \partial_t v_\zeta-\nu\partial_{zz}v_\zeta+\nu\zeta^2v_\zeta+\alpha u_\zeta+i\eta p_\zeta=f_{2,\zeta}\\ \partial_z p_\zeta-\beta\theta_\zeta=0\\ \partial_t\theta_\zeta-\nu\partial_{zz}\theta_\zeta+\nu\zeta^2\theta_\zeta+\gamma w_\zeta=f_{3,\zeta}\end{cases} \tag{51}$$
$$\text{with } w_\zeta(t,z)=-\textstyle\int_0^z(i\xi u_\zeta+i\eta v_\zeta),\;\; w_\zeta(a)=0,\;\; X_\zeta|_{z=0,z=a}=0$$

The equation above gives

$$p_\zeta(t,z)=p_\zeta(t,0)+\beta\int_0^z\theta_\zeta(t,z')\,dz' \tag{52}$$

So we set

$$\begin{array}{lll}\zeta=0\;: & c_0(t)=p_0(t,0) & ;\;\; q_0(t)=0\\ \zeta\neq0\;: & c_\zeta(t)=0 & ;\;\; q_\zeta(t)=p_\zeta(t,0)\end{array} \tag{53}$$

The spaces H^s_ζ. We define now the following space of functions of $z \in (0,a)$. For $f(z) = \sum_{k\in\mathbb{N}^*} f_k\, e_k(z)$, we set

$$\|f\|^2_{s,\zeta} = \sum_{k\in\mathbb{N}^*} (1+\nu k^2+\nu\zeta^2)^s |f_k|^2 \tag{54}$$

and we denote by H^s_ζ the Hilbert space associated with this latter norm. Similarly to lemma 1 for spaces $\mathcal{H}^s$ we have

Lemma 6

$$\begin{array}{rcl} -\frac{3}{2} < s < \frac{1}{2} \;\Rightarrow\; H^s_\zeta &=& H^s(0,a) \\ \frac{1}{2} < s < \frac{5}{2} \;\Rightarrow\; H^s_\zeta &=& \{f(z)\in H^s(0,a), f|_{z=0} = f|_{z=a} = 0\} \end{array} \tag{55}$$

And for $f(t,x,y,z) = \sum_{\zeta\in\mathbb{Z}^2} f_\zeta(t,z)\, e^{i\zeta.(x,y)}$ we obtain:

$$\|f(t)\|^2_{\mathcal{H}^s} = \sum_{\zeta\in\mathbb{Z}^2} \|f_\zeta(t,.)\|^2_{s,\zeta} \tag{56}$$

So that (16), (20) and (21) are equivalent to the following estimates, with C independent of ζ:

$$\|X_\zeta\|_{(L^2(\mathbb{R};H^{\sigma+2}_\zeta))^3} \;\leq\; C\,\|F_\zeta\|_{(L^2(\mathbb{R};H^\sigma_\zeta))^3} \tag{57}$$

and also for $q_\zeta = q_{1,\zeta} + q_{2,\zeta}$:

$$\begin{array}{lcl} -\frac{1}{2} < \sigma < \frac{1}{2} &:& \|q_\zeta\|_{L^2(\mathbb{R}_+;H^{\sigma+1}_\zeta)} \leq C\|F_\zeta\|_{L^2(\mathbb{R}_+;H^\sigma_\zeta)} \\ -\frac{3}{2} < \sigma < -\frac{1}{2} &:& \|q_{1,\zeta}\|_{H^{\sigma/2+1/4}(\mathbb{R}_+;H^1_\zeta)} + \|q_{2,\zeta}\|_{L^2(\mathbb{R}_+;H^{\sigma+1}_\zeta)} \leq C\|F_\zeta\|_{L^2(\mathbb{R}_+;H^\sigma_\zeta)} \end{array} \tag{58}$$

Case $\zeta = 0$.
In that case, p_0 vanishes from the first two equations of (51) , $w_0 = 0$ and (51) gives

$$\left\{\begin{array}{l} \partial_t u_0 - \nu\partial_{zz}u_0 - \alpha v_0 = f_{1,0} \\ \partial_t v_0 - \nu\partial_{zz}v_0 + \alpha u_0 = f_{2,0} \\ \partial_t\theta_0 - \nu\partial_{zz}\theta_0 = f_{3,0} \\ p_0(t,z) = c_0(t) + \beta\int_0^z \theta_0(t,z')\,dz' \end{array}\right. \tag{59}$$
$$\text{with } X_0|_{z=0,z=a} = 0$$

So that classical results on the heat equation give:

$$\begin{array}{rcl} \|X_0\|_{(L^2(\mathbb{R};H^{\sigma+2}_0))^3} &\leq& C\,\|F_0\|_{(L^2(\mathbb{R};H^\sigma_0))^3} \\ \|\partial_t X_0\|_{(L^2(\mathbb{R};H^\sigma_0))^3} &\leq& C\,\|F_0\|_{(L^2(\mathbb{R};H^\sigma_0))^3} \end{array} \tag{60}$$

Moreover $q_0(t) = 0$ and estimates (20) and (21) are immediate.
In the sequel, we assume that $\zeta \neq 0$.
For the pressure p we have then

$$p_\zeta(t,z) = q_\zeta(t) + \beta \int_0^z \theta_\zeta(t,z')\,dz' \tag{61}$$

so that existence and unicity for u, v and θ give those of p (up to the constant $c(t)$) and(18).

Fourier-Laplace transform in time. For $f(t,z) \in L^2(\mathbb{R}; H^\sigma_\zeta)$ with support in $\{t \geq 0\}$, we denote by $\hat{f}(\tau)$ its Fourier-Laplace transform:

$$\hat{f}(\tau,z) = \int_0^{+\infty} e^{-it\tau} f(t,z)\,dt \tag{62}$$

It is clear that $\hat{f}$ is holomorphic in $\{\tau \in \mathbb{C}, \Im(\tau) < 0\}$ and satisfies

$$\int_{-\infty}^{+\infty} \|\hat{f}(\tau)\|^2_{H^\sigma_\zeta}\,d\tau \;=\; C_0\,\|f\|^2_{L^2(\mathbb{R}_+;H^\sigma_\zeta)} \tag{63}$$

From (51), for a given $\zeta \neq 0$, for $u_\zeta, v_\zeta \in L^2(\mathbb{R}; H^\sigma_\zeta)$ we have

$$p_\zeta \in L^2(\mathbb{R}_+; H^\sigma_\zeta) + \partial_t L^2(\mathbb{R}_+; H^{\sigma+2}_\zeta) \tag{64}$$

where $\partial_t L^2(\mathbb{R}_+; H^{\sigma+2}_\zeta) = \{q, \exists \widetilde{q} \in L^2(\mathbb{R}_+; H^{\sigma+2}_\zeta), q = \partial_t \widetilde{q}\}$. Thus the Fourier-Laplace transform of p_ζ is well-defined.

Introduction of the parameters. Let $\mathbb{S}$ be the subset of $\mathbb{C}$ defined by:

$$\mathbb{S} = \{-\delta_2 - \mu_1 + i\mu_2, \text{ with } (\mu_1,\mu_2) \in \mathbb{R}^2 \text{ and } |\mu_2| \geq \frac{1}{\delta_1}\mu_1\} \tag{65}$$

with $\delta_1 > 0$ small enough that $\mathbb{S} \cap \mathbb{V}_P = \emptyset$ (which is possible from lemma 3) and

$$\delta_2 < \delta_3 = \min(\frac{\nu\pi^2}{2a^2}, \frac{\nu}{2}) \tag{66}$$

For $\zeta^2 \in \mathbb{Z}^2 \setminus 0$ and $\lambda \in \mathbb{S}$, we set

$$\lambda = i\tau, \quad \omega^2 = \lambda + \nu\zeta^2 \tag{67}$$

so that

$$\left.\begin{array}{l}\lambda \in \mathbb{S}\\ \zeta \in \mathbb{Z}^2 \setminus 0\end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} w^2 \neq 0\\ \lambda + \delta_3 \neq 0\\ \nu\zeta^2 - \delta_3 > 0\end{array}\right. \tag{68}$$

Keeping the same notations for functions and their Fourier-Laplace transform, we obtain that (51) is equivalent to the following equations, with parameter $(\zeta^2, \lambda) \in \mathbb{Z}^2 \setminus 0 \times \mathbb{S}$:

$$\left\{ \begin{array}{l} (\omega^2 - \nu\partial_{zz})u_\zeta - \alpha v_\zeta + i\xi p_\zeta = f_{1,\zeta} \\ (\omega^2 - \nu\partial_{zz})v_\zeta + \alpha u_\zeta + i\eta p_\zeta = f_{2,\zeta} \\ \partial_z p_\zeta - \beta\theta_\zeta = 0 \\ (\omega^2 - \nu\partial_{zz})\theta_\zeta + \gamma w_\zeta = f_{3,\zeta} \end{array} \right. \\ \text{with } w_\zeta(z) = -\int_0^z (i\xi u_\zeta + i\eta v_\zeta) \\ \text{and } w_\zeta(a) = 0,\ X_\zeta|_{z=0,z=a} = 0 \tag{69}$$

For ζ and λ given $\mathbb{Z}^2 \setminus 0$ and $\mathbb{S}$, (69) is a differential system as a function of $z \in (0,a)$ with data $F_\zeta \in H^\sigma(0,a)$ and unknown $X_\zeta \in H^\sigma(0,a)$, $X_\zeta|_{z=0,a} = 0$. Therefore, unicity is clear. Indeed, the third equation of (69) gives $p_\zeta \in H^{\sigma+3}(0,a)$, so that if $F = 0$ we obtain $u_\zeta, v_\zeta \in H^{\sigma+5}(0,a)$, thus $w \in H^{\sigma+6}(0,a)$, then $\theta \in H^{\sigma+8}(0,a)$. In particular we have $X_\zeta \in H_0^1(0,a)$, and the spectral result $\mathbb{S} \cap \mathbb{V}_P = \emptyset$ gives $\hat{X}_\zeta = 0$ for all $(\lambda, \zeta) \in \mathbb{Z}^2 \setminus 0 \times \mathbb{S}$. Then $X = 0$ and $p = 0$ up to a function of time.
It remains to prove existence and estimates for the solutions of (69), which will give (57) and (58) thanks to (63).

3.2 Preliminary results for the parameters

We set

$$\langle\zeta\rangle = 1 + |\zeta|, \quad \langle\omega\rangle^2 = |\lambda| + \langle\zeta\rangle^2 \tag{70}$$

The following lemma will be useful:

Lemma 7 *There exists a constant C such that, for all $\lambda \in \mathbb{S}$, $\zeta \in \mathbb{Z}^2 \setminus 0$ and $k \in \mathbb{N}^*$:*

$$|\omega^2| \geq C\,\langle\omega\rangle^2 \geq C\,(1 + |\lambda| + \zeta^2) \tag{71}$$

and

$$|\omega^2 + \frac{\nu k^2 \pi^2}{a^2}| \geq C\,(\langle\omega\rangle^2 + k^2). \tag{72}$$

This lemma is easily obtained from the straightforward following lemma:

Lemma 8 *Let C_1 and C_2 be two closed cones of $\mathbb{R}^n$. We assume the distance between C_1 and C_2 to be non-zero, ie there exists a constant $d > 0$ such that*

$$\forall x \in C_1, \forall y \in C_2, \quad \|x\| = \|y\| = 1 \Rightarrow \|x - y\| \geq d \tag{73}$$

Then there exists a constant $C > 0$ such that

$$\forall x \in C_1, \forall y \in C_2, \quad \|x - y\| \geq C\,(\|x\| + \|y\|) \tag{74}$$

The function $M_\sigma(\lambda,\zeta)$. For $(\lambda,\zeta)\in\mathbb{S}\times\mathbb{Z}^2\backslash 0$ we set:

$$M_\sigma(\lambda,\zeta)=\Big(\sum_{k\in\mathbb{N}^*}\frac{1}{k^2(k^4+\langle\omega\rangle^4)(k^2+\langle\zeta\rangle^2)^\sigma}\Big)^{1/2} \tag{75}$$

which is well-defined for $\sigma>-\frac{5}{2}$.
We shall use the following notation

$$A(\lambda,\zeta)\sim B(\lambda,\zeta)\quad\Leftrightarrow\quad\left\{\begin{array}{l}\exists C_1,C_2>0,\ \forall(\lambda,\zeta)\in\mathbb{S}\times\mathbb{Z}^2\backslash 0,\\ C_1\,B(\lambda,\zeta)\ \le\ A(\lambda,\zeta)\ \le\ C_2\,B(\lambda,\zeta)\end{array}\right. \tag{76}$$

Lemma 9 *For $\sigma\in]-\frac{5}{2},\frac{3}{2}[$, $\sigma\neq-\frac{1}{2}$, the following hold true:*

$$\begin{array}{lll}\sigma>-\frac{1}{2}: & M_\sigma\ \sim & \dfrac{\langle\zeta\rangle^{-\sigma}}{\langle\omega\rangle^2}\\ \sigma<-\frac{1}{2}: & M_\sigma\ \sim & \dfrac{\langle\zeta\rangle^{-\sigma}}{\langle\omega\rangle^2}\mathbb{1}_{\langle\zeta\rangle\ge\langle\omega\rangle^\kappa}+\dfrac{\langle\omega\rangle^{-\sigma}}{\langle\omega\rangle^{\frac{5}{2}}}\mathbb{1}_{\langle\zeta\rangle\le\langle\omega\rangle^\kappa}\,,\qquad \textit{with } \kappa=\frac{2\sigma+1}{2\sigma}\end{array} \tag{77}$$

Proof. First we can write:

$$\begin{array}{rl}M_\sigma^2=\sum\frac{(k^2+\langle\zeta\rangle^2)^{-\sigma}}{k^2\,(\langle\omega\rangle^4+k^4)}\ \sim & \big[\int_1^{\langle\zeta\rangle}\frac{(x^2+\langle\zeta\rangle^2)^{-\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx+\int_{\langle\zeta\rangle}^{\langle\omega\rangle}\frac{(x^2+\langle\zeta\rangle^2)^{-\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx\\ & \quad+\int_{\langle\omega\rangle}^{\infty}\frac{(x^2+\langle\zeta\rangle^2)^{-\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx\big]\end{array} \tag{78}$$

We have immediately:

$$\begin{array}{rcll}I_1&=&\int_1^{\langle\zeta\rangle}\frac{(x^2+\langle\zeta\rangle^2)^{-\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx&\sim\ \frac{\langle\zeta\rangle^{-2\sigma}}{\langle\omega\rangle^4}\int_1^{\langle\zeta\rangle}\frac{dx}{x^2}\\ &&&\sim\ \frac{\langle\zeta\rangle^{-2\sigma}}{\langle\omega\rangle^4}\\ I_2&=&\int_{\langle\zeta\rangle}^{\langle\omega\rangle}\frac{(x^2+\langle\zeta\rangle^2)^{-\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx&\sim\ \int_{\langle\zeta\rangle}^{\langle\omega\rangle}\frac{x^{-2\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx\\ &&&\sim\ \frac{1}{\langle\omega\rangle^4}\int_{\langle\zeta\rangle}^{\langle\omega\rangle}\frac{dx}{x^{2\sigma+2}}\\ I_3&=&\int_{\langle\omega\rangle}^{\infty}\frac{(x^2+\langle\zeta\rangle^2)^{-\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx&\sim\ \int_{\langle\omega\rangle}^{\infty}\frac{x^{-2\sigma}}{x^2\,(\langle\omega\rangle^4+x^4)}\,dx\\ &&&\sim\ \frac{\langle\omega\rangle^{-2\sigma}}{\langle\omega\rangle^5}\int_1^\infty\frac{u^{-2\sigma}}{u^2\,(1+u^4)}\,du\\ &&&\sim\ \frac{\langle\omega\rangle^{-2\sigma}}{\langle\omega\rangle^5}\end{array} \tag{79}$$

Then there are two cases:
⋆ Case $\sigma>-\frac{1}{2}$, with $\sigma<0$: then $\frac{\langle\omega\rangle^{-2\sigma}}{\langle\omega\rangle}$ is bounded and $\langle\zeta\rangle^{-2\sigma}\ge 1$ thus

$$\frac{\langle\omega\rangle^{-2\sigma}}{\langle\omega\rangle^5}\ \le C\frac{1}{\langle\omega\rangle^4}\ \le\ C\frac{\langle\zeta\rangle^{-2\sigma}}{\langle\omega\rangle^4}\quad\Rightarrow\quad I_3\ \le\ CI_1 \tag{80}$$

If $\sigma > -\frac{1}{2}$, with $\sigma > 0$, then $\langle\omega\rangle^{-2\sigma} \leq \langle\zeta\rangle^{-2\sigma}$ thus $I_3 \leq CI_1$.
Moreover, if $\sigma > -\frac{1}{2}$ we have $2\sigma + 2 > 1$ then

$$\int_{\langle\zeta\rangle}^{\langle\omega\rangle} \frac{dx}{x^{2\sigma+2}} \leq C\langle\zeta\rangle^{-2\sigma-1} \quad \Rightarrow \quad I_2 \leq CI_1 \tag{81}$$

It follows that (77) holds true for $\sigma > -\frac{1}{2}$.
$\star$ Case $\sigma < -\frac{1}{2}$: we have

$$\int_{\langle\zeta\rangle}^{\langle\omega\rangle} \frac{dx}{x^{2\sigma+2}} \leq C\langle\omega\rangle^{-2\sigma-1} \quad \Rightarrow \quad I_2 \leq CI_3 \tag{82}$$

In order to compare I_1 and I_3 we introduce the following critical exponent:

$$\kappa = \frac{2\sigma+1}{2\sigma} \quad \Leftrightarrow \quad -2\sigma\kappa = -2\sigma - 1 \tag{83}$$

If $\sigma \in]-\frac{5}{2}, -\frac{1}{2}[$, then $\kappa \in]0, \frac{4}{5}[$. We have again two cases:
$*$ If $(\langle\zeta\rangle, \langle\omega\rangle) \in \{\langle\zeta\rangle \leq \langle\omega\rangle^{\kappa}\}$, then

$$\langle\zeta\rangle^{-2\sigma} \leq \langle\omega\rangle^{-2\sigma\kappa} = \langle\omega\rangle^{-2\sigma-1} \quad \Rightarrow \quad I_1 \leq C\, I_3 \tag{84}$$

$*$ If $(\langle\zeta\rangle, \langle\omega\rangle) \in \{\langle\zeta\rangle \geq \langle\omega\rangle^{\kappa}\}$, then

$$\langle\zeta\rangle^{-2\sigma} \geq \langle\omega\rangle^{-2\sigma\kappa} = \langle\omega\rangle^{-2\sigma-1} \quad \Rightarrow \quad I_3 \leq C\, I_1 \tag{85}$$

And (77) follows, for $\sigma < -\frac{1}{2}$.
□
We have also

Corollary 2 *If* $\sigma \in]-\frac{1}{2}, \frac{1}{2}[$ *then*

$$\langle\omega\rangle^2 M_\sigma M_{-\sigma} \quad \sim \quad \langle\omega\rangle^{-2} \tag{86}$$

If $\sigma \in]-\frac{3}{2}, -\frac{1}{2}[$ *then*

$$\langle\omega\rangle^2 M_\sigma M_{-\sigma} \quad \leq \quad \langle\omega\rangle^{-1} \tag{87}$$

and

$$\langle\omega\rangle^2 M_\sigma M_{-\sigma-2} \quad \leq \quad C\, \mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa} + \langle\omega\rangle^{-1}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa} \tag{88}$$

Proof. If $\sigma \in]-\frac{1}{2},\frac{1}{2}[$, (86) is immediately obtained from lemma 9. If $\sigma \in]-\frac{3}{2},-\frac{1}{2}[$ then

$$\begin{array}{rcl}\langle\omega\rangle^2 M_\sigma M_{-\sigma} & \sim & \langle\omega\rangle^2\big(\frac{\langle\zeta\rangle^{-\sigma}}{\langle\omega\rangle^2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\frac{\langle\omega\rangle^{-\sigma}}{\langle\omega\rangle^{5/2}}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}\big)\frac{\langle\zeta\rangle^{\sigma}}{\langle\omega\rangle^2}\\ & \sim & \langle\omega\rangle^{-2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\zeta\rangle^{\sigma}\langle\omega\rangle^{-\sigma-5/2}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}\\ & \leq & \langle\omega\rangle^{-2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\omega\rangle^{-\sigma-5/2}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}\\ & \leq & \langle\omega\rangle^{-1}\end{array}\tag{89}$$

because $\langle\zeta\rangle^\sigma < 1$ as $\sigma < 0$ and $\langle\omega\rangle^{-\sigma-5/2} < \langle\omega\rangle^{-1}$ as $-\sigma - 5/2 < -1$.
Finally we have

$$\begin{array}{rcl}\langle\omega\rangle^2 M_\sigma M_{-\sigma-2} & \sim & \langle\omega\rangle^2(\langle\zeta\rangle^{-\sigma}\langle\omega\rangle^{-2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\omega\rangle^{-\sigma-5/2}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa})\\ & & \quad(\langle\zeta\rangle^{\sigma+2}\langle\omega\rangle^{-2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\omega\rangle^{\sigma-1/2}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa})\\ & \sim & \langle\zeta\rangle^2\langle\omega\rangle^{-2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\omega\rangle^{-1}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}\\ & \leq & C\,\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\omega\rangle^{-1}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}\end{array}\tag{90}$$

□

3.3 The uncoupled system

Having disposed of these preliminary steps, we can now study the uncoupled system (69) ie with $\alpha = \beta = \gamma = 0$:

$$\left\{\begin{array}{l}(\omega^2-\nu\partial_{zz})u_\zeta+i\xi p_\zeta=f_{1,\zeta}\\(\omega^2-\nu\partial_{zz})v_\zeta+i\eta p_\zeta=f_{2,\zeta}\\\partial_z p_\zeta=0\\(\omega^2-\nu\partial_{zz})\theta_\zeta=f_{3,\zeta}\end{array}\right.\tag{91}$$
$$\text{with } \int_0^a(\xi u_\zeta+\eta v_\zeta)=0 \text{ and } X_\zeta|_{z=0,z=a}=0$$

Notations. We denote for short:

- $p_{\zeta,0} = p_\zeta(z = 0)$;
- $\|.\|_{\sigma,\zeta}$ stands for $\|.\|_{H^\sigma_\zeta}$, $\|.\|_{(H^\sigma_\zeta)^2}$ and $\|.\|_{(H^\sigma_\zeta)^3}$;
- $(H^{\sigma+2}_\zeta)^2_{\rm div}$ stands for $\{(u,v)\in(H^{\sigma+2}_\zeta)^2, \int_0^a \xi u+\eta v=0\}$.

The next theorem is the core of the proof:

Theorem 2 *Let* $(\lambda,\zeta)\in\mathbb{S}\times\mathbb{Z}^2\backslash 0$. *The operator*

$$\begin{array}{cccc} & (H_\zeta^{\sigma+2})^2_{div}\times\mathbb{C}\times H_\zeta^{\sigma+2} & \to & (H_\zeta^\sigma)^3 \\ \mathcal{L}_0: & (u,v,p_0,\theta) & \mapsto & \begin{bmatrix} (\omega^2-\nu\partial_{zz})u+i\xi p_0 \\ (\omega^2-\nu\partial_{zz})v+i\eta p_0 \\ (\omega^2-\nu\partial_{zz})\theta \end{bmatrix} = \begin{bmatrix} f_1\\ f_2\\ f_3\end{bmatrix} \end{array} \tag{92}$$

is continuous and bijective. Moreover $Y=(u,v)$ *splits in* Y_1+Y_2 *and the following estimates hold:*

$$\begin{array}{lrcl} (a) & |\zeta p_0| & \leq & C\langle\omega\rangle^2 M_\sigma\|F\|_{\sigma,\zeta} \\ (b) & M_{-\sigma-2}[M_{-\sigma}]^{-1}\|Y_1\|_{\sigma,\zeta}+\|Y_1\|_{\sigma+2,\zeta} & \leq & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\|F\|_{\sigma,\zeta} \\ (c) & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta}+\|Y_2\|_{\sigma+2,\zeta} & \leq & C\|F\|_{\sigma,\zeta} \\ (d) & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta}+\|\theta\|_{\sigma+2,\zeta} & \leq & C\|f_3\|_{\sigma,\zeta} \end{array} \tag{93}$$

with $(f_1,f_2)=F$.

Before proving this theorem, we state an elementary but useful remark:

Remark 6 *The eigenvalues of the operator* $\omega^2-\nu\partial_{zz}$ *are eigenvalues of* P. *Moreover, for all* $\lambda\notin\mathbb{V}_P$, *the operator* $\omega^2-\nu\partial_{zz}$ *is continuous and bijective from* $H_\zeta^{\sigma+2}$ *to* H_ζ^σ, *for all* $\sigma\in]-\frac{3}{2},\frac{1}{2}[$.

Proof of theorem2. Continuity is easy, as is injectivity (because $\mathbb{S}\cap\mathbb{V}_P=\emptyset$).
We first examine the θ part. Surjectivity is clear. Then θ satisfies a heat equation for which we can write a maximal estimate, which is exactly (93,d): indeed let $f\in H_\zeta^\sigma$, with $f=\sum_{k\in\mathbb{N}}f_ke_k(z)$. Let g be the solution of $(\omega^2-\nu\partial_{zz})g=f$ with homogeneous Dirichlet boundary conditions. Then $g\in H^{\sigma+2}(0,a)$, with $\sigma+2\in]\frac{1}{2},\frac{5}{2}[$, and $g(0)=g(a)=0$, hence $g\in H_\zeta^{\sigma+2}$ according to lemma 6. Thus $g=\sum_k g_ke_k$ and we have:

$$(\omega^2+\frac{\nu k^2\pi^2}{a^2})g_k=f_k \tag{94}$$

with $\omega^2+\frac{\nu k^2\pi^2}{a^2}\neq 0$.
Using lemma 7 we obtain easily:

$$\begin{array}{rcl} \langle\omega\rangle^4\|g\|^2_{\sigma,\zeta} & = & \langle\omega\rangle^4\sum_k(1+\nu\zeta^2+\nu k^2)^\sigma\dfrac{|f_k|^2}{|\omega^2+\frac{\nu k^2\pi^2}{a^2}|^2} \\ & \leq & C\langle\omega\rangle^4\sum_k(1+\nu\zeta^2+\nu k^2)^\sigma\dfrac{|f_k|^2}{\langle\omega\rangle^4+k^4} \quad \leq \quad C\|f\|^2_{\sigma,\zeta} \end{array} \tag{95}$$

and

$$
\begin{aligned}
\|g\|^2_{\sigma+2,\zeta} &\leq \sum_k (1+\nu\zeta^2+\nu k^2)^\sigma |f_k|^2 \frac{(1+\nu\zeta^2+\nu k^2)^2}{|\omega^2+\frac{\nu k^2\pi^2}{a^2}|^2} \\
&\leq C\sum_k (1+\nu\zeta^2+\nu k^2)^\sigma |f_k|^2 \frac{\langle\zeta\rangle^4+k^4}{\langle\omega\rangle^4+k^4} \quad \leq \quad C\|f\|^2_{\sigma,\zeta}
\end{aligned}
\tag{96}
$$

With $f = f_{3,\zeta} \in H^\sigma_\zeta$ and $\theta_\zeta = g$ we get (93,d).

We know examine $\mathcal{L}_0$ restricted to $(u, v, p_0) \in (H^{\sigma+2}_\zeta)^2_{\text{div}} \times \mathbb{C}$. To establish surjectivity, we first evaluate (explicitly) the constant p_0, we then invert the operator $(\omega^2 - \nu\partial_{zz})$ (which is possible according to remark 6). Thus it is sufficient to evaluate p_0 and prove estimates (93).

With (92) we obtain:

$$
u = (\omega^2-\nu\partial_{zz})^{-1}[f_1 - i\xi p_0], \quad v = (\omega^2-\nu\partial_{zz})^{-1}[f_1 - i\eta p_0] \tag{97}
$$

then we evaluate p_0 thanks to $\int_0^a i\xi u + i\eta v = 0$:

$$
\begin{aligned}
& \textstyle\int_0^a i\xi(\omega^2-\nu\partial_{zz})^{-1}[f_1 - i\xi p_0] + i\eta(\omega^2-\nu\partial_{zz})^{-1}[f_1 - i\eta p_0]\, dz = 0 \\
\Leftrightarrow\ & \textstyle\int_0^a (\omega^2-\nu\partial_{zz})^{-1}[(\xi^2+\eta^2)p_0]\, dz = -\int_0^a (\omega^2-\nu\partial_{zz})^{-1}[i\xi f_1 + i\eta f_2]\, dz \\
\Leftrightarrow\ & \textstyle\zeta^2 p_0 \int_0^a (\omega^2-\nu\partial_{zz})^{-1}[1]\, dz = -\int_0^a (\omega^2-\nu\partial_{zz})^{-1}[i\xi f_1 + i\eta f_2]\, dz
\end{aligned}
\tag{98}
$$

We will prove further the following lemma:

Lemma 10 *There exist constants C_1 and C_2, independent of $\omega^2 = \lambda + \nu\zeta$, such that, for all $(\lambda, \zeta) \in \mathbb{S} \times \mathbb{Z} \setminus 0$,*

$$
\frac{C_1}{\langle\omega\rangle^2} \leq \int_0^a (\lambda - \nu\Delta)^{-1}(1)\, dz \leq \frac{C_2}{\langle\omega\rangle^2} \tag{99}
$$

Thus p_0 is well-defined by the following formula:

$$
\zeta^2 p_0 = -\Big[\int_0^a (\lambda-\nu\Delta)^{-1}(1)\, dz\Big]^{-1} \int_0^a (\omega^2-\nu\partial_{zz})^{-1}[i\xi f_1 + i\eta f_2]\, dz \tag{100}
$$

Let us now estimate $\int_0^a (\omega^2-\nu\partial_{zz})^{-1}[i\xi f_1 + i\eta f_2]\, dz$. Let $f = \sum_k f_k e_k \in H^\sigma_\zeta$. As previously, using lemma 7, we have

$$
\begin{aligned}
|\textstyle\int_0^a (\omega^2-\nu\partial_{zz})^{-1} f| &= |\textstyle\int_0^a \sum \dfrac{f_k}{\omega^2+\frac{\nu k^2\pi^2}{a^2}} e_k(z)\, dz| \\
&\leq C\sum \frac{|f_k|}{k\,(\langle\omega\rangle^2+k^2)} \\
&\leq C\|f\|_{\sigma,\zeta}\Big[\sum \frac{(1+\nu k^2+\nu|\zeta|^2)^{-\sigma}}{k^2\,(\langle\omega\rangle^2+k^2)^2}\Big]^{1/2} \\
&\leq C\|f\|_{\sigma,\zeta} M_\sigma(\lambda,\zeta)
\end{aligned}
\tag{101}
$$

Using (99), (100) and (101) we obtain:

$$\zeta^2\,|p_0|\;\le\;C\,\langle\omega\rangle^2\,M_\sigma(\lambda,\zeta)\,|\zeta|\,\|F\|_{\sigma,\zeta} \tag{102}$$

and (93,a) and surjectivity follows.
To establish (93,b,c) we use (97):

$$u=(\omega^2-\nu\partial_{zz})^{-1}[f_1]-i\xi p_0(\omega^2-\nu\partial_{zz})^{-1}[1]=u_2+u_1 \tag{103}$$

Define

$$\begin{aligned} Y_1 &= -i\begin{bmatrix}\xi\\ \eta\end{bmatrix}p_0(\omega^2-\nu\partial_{zz})^{-1}[1]\\ Y_2 &= (\omega^2-\nu\partial_{zz})^{-1}\begin{bmatrix}f_1\\ f_2\end{bmatrix}\end{aligned} \tag{104}$$

Like θ_ζ, Y_2 satisfies the maximal estimate of the heat equation, namely:

$$\langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta}+\|Y_2\|_{\sigma+2,\zeta}\le C\|F\|_{\sigma,\zeta} \tag{105}$$

We now turn to Y_1:

$$\|Y_1\|_{\sigma+2,\zeta}=|\zeta|\,|p_0|\,\|(\omega^2-\nu\partial_{zz})^{-1}[1]\|_{\sigma+2,\zeta} \tag{106}$$

As previously, the constant function 1 stays in H^σ_ζ and $1=\sum_k a_k e_k$, with $a_k=C_0\frac{1}{k}$ for k odd and $a_k=0$ otherwise, and we obtain:

$$\|(\omega^2-\nu\partial_{zz})^{-1}[1]\|_{\sigma+2,\zeta}\;\le\;C\Big[\sum_k\frac{(\langle\zeta\rangle^2+k^2)^{\sigma+2}}{k^2(\langle\omega\rangle^4+k^4)}\Big]^{1/2}\;=\;CM_{-\sigma-2}(\lambda,\zeta) \tag{107}$$

($M_{-\sigma-2}$ is well-defined for $\sigma\in]-\frac{3}{2},\frac{1}{2}[$)
Likewise we get:

$$\|(\omega^2-\nu\partial_{zz})^{-1}[1]\|_{\sigma,\zeta}\;\le\;CM_{-\sigma}(\lambda,\zeta) \tag{108}$$

($M_{-\sigma}$ is well-defined for $\sigma\in]-\frac{3}{2},\frac{1}{2}[$)
which completes the proof.
□

Proof of lemma 10. Straightforward calculation gives:

$$\int_0^a(\lambda-\nu\Delta)^{-1}(1)\,dz=\frac{a}{\omega^2}\Big[1+2\frac{1-\cosh(\frac{\omega a}{\sqrt{\nu}})}{\frac{\omega a}{\sqrt{\nu}}\sinh(\frac{\omega a}{\sqrt{\nu}})}\Big] \tag{109}$$

where ω is the square root of ω^2 with positive real part. Let us define

$$\mathcal{N}(\chi) = 1 + 2\frac{1-\cosh(\chi)}{\chi\sinh(\chi)} \tag{110}$$

with $\chi = \frac{\omega a}{\sqrt{\nu}}$. We have $\omega^2 = \lambda + \nu\zeta$, thus according to (65), (66) and (68) ω and χ stays in $\mathbb{B}$ with

$$\mathbb{B} = \{\mu_1 + i\mu_2,\ \text{with } (\mu_1,\mu_2) \in \mathbb{R}_+ \times \mathbb{R} \text{ and } |\mu_2| \leq (1+\delta_5)\mu_1\} \setminus \delta_4 B_1 \tag{111}$$

where δ_4 and δ_5 are small and B_1 is the open unity disk.
The singularities of $\mathcal{N}$ are zeros of sinh, so that $\mathcal{N}$ is holomorphic in $\mathbb{B}$. Moreover, $\mathcal{N}$ has no zero in $\mathbb{B}$, because $\mathcal{L}_0$ is injective for $\omega \in \mathbb{B}$. We have besides

$$\lim_{|\chi|\to+\infty} \mathcal{N}(\chi) = 1 \tag{112}$$

Hence there exist constants C_1 and C_2, independent of ω, such that for all $\chi \in \mathbb{B}$

$$\frac{C_1}{1+|\chi|^2} \leq \frac{|\mathcal{N}(\chi)|}{|\chi|^2} \leq \frac{C_2}{1+|\chi|^2} \tag{113}$$

Estimate (99) easily follows from (113) and lemma 7.
□

3.4 The coupled system

We now turn to the case $\alpha, \beta, \gamma \neq 0$ and system (69). Let us define the corresponding perturbation operator $\mathcal{L}_1$:

$$\begin{array}{cccc} & (H_\zeta^{\sigma+2})^2_{\mathrm{div}} \times \mathbb{C} \times H_\zeta^{\sigma+2} & \to & (H_\zeta^\sigma)^3 \\ \mathcal{L}_1 : & (u,v,p_0,\theta) & \mapsto & \begin{bmatrix} -\alpha v + i\xi\beta\int_0^z \theta \\ \alpha u + i\eta\beta\int_0^z \theta \\ -\gamma\int_0^z (i\xi u + i\eta v) \end{bmatrix} \end{array} \tag{114}$$

such that $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where $\mathcal{L}_0$ is given by (92), corresponds to system (69). We have the

Theorem 3 *Let $(\lambda,\zeta) \in \mathbb{S} \times \mathbb{Z}^2\backslash 0$. The following operator*

$$\begin{array}{cccc} & (H_\zeta^{\sigma+2})^2_{div} \times \mathbb{C} \times H_\zeta^{\sigma+2} & \to & (H_\zeta^\sigma)^3 \\ \mathcal{L} : & (u,v,p_0,\theta) & \mapsto & (\mathcal{L}_0 + \mathcal{L}_1)(u,v,p_0,\theta) = F \end{array} \tag{115}$$

is continuous and bijective. Moreover $Y = (u, v)$ splits in $Y_1 + Y_2$ and the following estimates hold true:

$$
\begin{array}{llcl}
(a) & |\zeta p_0| & \leq & C\langle\omega\rangle^2 M_\sigma \|F\|_{\sigma,\zeta} \\
(b) & M_{-\sigma-2}[M_{-\sigma}]^{-1}\|Y_1\|_{\sigma,\zeta} + \|Y_1\|_{\sigma+2,\zeta} & \leq & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\|F\|_{\sigma,\zeta} \\
(c) & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta} + \|Y_2\|_{\sigma+2,\zeta} & \leq & C\|F\|_{\sigma,\zeta} \\
(d) & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta} + \|\theta\|_{\sigma+2,\zeta} & \leq & C\|F\|_{\sigma,\zeta}
\end{array}
\tag{116}
$$

Proof. Let $(\lambda, \zeta) \in \mathbb{S} \times \mathbb{Z}^2\backslash 0$. To prove that $\mathcal{L}$ is an isomorphism, we first state that the image of $\mathcal{L}_1$ is included in a compact subspace of $(H^\sigma_\zeta)^3$. Indeed, if $(u, v, p_0, \theta) \in (H^{\sigma+2}_\zeta)^2_{\text{div}} \times \mathbb{C} \times H^{\sigma+2}_\zeta$, then

$$
\begin{array}{rcl}
-\alpha v + i\xi\beta \int_0^z \theta & \in & H^{\sigma+2}(0, a) \\
\alpha u + i\eta\beta \int_0^z \theta & \in & H^{\sigma+2}(0, a) \\
-\gamma \int_0^z (i\xi u + i\eta v) & \in & H^{\sigma+3}(0, a)
\end{array}
\tag{117}
$$

and $(H^{\sigma+2}(0, a))^2 \times H^{\sigma+3}(0, a)$ is a compact subspace of $(H^\sigma_\zeta)^3$. From Fredholm theory, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ is an isomorphism if and only if its kernel is trivial. Let (u, v, p_0, θ) be in the kernel of $\mathcal{L}$, ie

$$
\begin{array}{l}
\left\{
\begin{array}{l}
(\omega^2 - \nu\partial_{zz})u = \alpha v - i\xi p \\
(\omega^2 - \nu\partial_{zz})v = -\alpha u - i\eta p \\
\partial_z p - \beta\theta = 0 \\
(\omega^2 - \nu\partial_{zz})\theta = -\gamma w
\end{array}
\right. \\
\text{with } w(z) = -\int_0^z (i\xi u + i\eta v) \\
\text{and } w(a) = 0,\ X|_{z=0,z=a} = 0
\end{array}
\tag{118}
$$

Thus u, v and θ are smooth and $0 \notin \mathbb{V}_P$ gives $(u, v, \theta) = 0$ and then $p_0 = 0$. Let us turn now to estimates (116). We write

$$
\mathcal{L}(u, v, p_0, \theta) = F \quad \Leftrightarrow \quad \mathcal{L}_0(u, v, p_0, \theta) = F - \mathcal{L}_1(u, v, p_0, \theta) \tag{119}
$$

and we use theorem 2. We get:

$$
\begin{array}{llcl}
(a) & |\zeta p_0| & \leq & C\langle\omega\rangle^2 M_\sigma \big[\|F\|_{\sigma,\zeta} + \|Y\|_{\sigma,\zeta} + |\zeta| \| \int_0^z \theta\, dz\|_{\sigma,\zeta}\big] \\
(b) & \frac{M_{-\sigma-2}}{M_{-\sigma}}\|Y_1\|_{\sigma,\zeta} + \|Y_1\|_{\sigma+2,\zeta} & \leq & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big[\|F\|_{\sigma,\zeta} + \|Y\|_{\sigma,\zeta} + |\zeta| \| \int_0^z \theta\, dz\|_{\sigma,\zeta}\big] \\
(c) & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta} + \|Y_2\|_{\sigma+2,\zeta} & \leq & C\big[\|F\|_{\sigma,\zeta} + \|Y\|_{\sigma,\zeta} + |\zeta| \| \int_0^z \theta\, dz\|_{\sigma,\zeta}\big] \\
(d) & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta} + \|\theta\|_{\sigma+2,\zeta} & \leq & C\big[\|F\|_{\sigma,\zeta} + |\zeta| \| \int_0^z Y\, dz\|_{\sigma,\zeta}\big]
\end{array}
\tag{120}
$$

We will prove further the following lemma:

Lemma 11 *Let $s \in]-\frac{3}{2}, \frac{1}{2}[$, $s \neq -\frac{1}{2}$ and $\varphi \in H^s_\zeta \cap H^{s+2}_\zeta$. The function $\phi(z) = \int_0^z \varphi(z')\,dz'$ stays in H^s_ζ.*
Besides, if $s \in]-\frac{1}{2}, \frac{1}{2}[$, we get

$$\|\phi\|_{s,\zeta} \le C\|\varphi\|_{s,\zeta} \tag{121}$$

where the constant C is independent of ζ and ω.
If $s \in]-\frac{3}{2}, -\frac{1}{2}[$, we get

$$\begin{array}{lcl} \|\phi\|_{s,\zeta} & \le & C_2(\omega,\zeta)\left(\langle\omega\rangle^2\|\varphi\|_{s,\zeta} + \|\varphi\|_{s+2,\zeta}\right) \\ \|\phi\|_{s,\zeta} & \le & C_1(\omega,\zeta)\left(\frac{M_{-s-2}}{M_{-s}}\|\varphi\|_{s,\zeta} + \|\varphi\|_{s+2,\zeta}\right) \end{array} \tag{122}$$

with:

$$\begin{array}{lll} (i) & \langle\zeta\rangle\, C_2(\omega,\zeta) & \overset{\langle\omega\rangle\to\infty}{\longrightarrow} 0 \\ (ii) & \langle\zeta\rangle\, C_1(\omega,\zeta)\, \mathbb{1}_{\langle\zeta\rangle \ge \langle\omega\rangle^\kappa} & \overset{\langle\omega\rangle\to\infty}{\longrightarrow} 0 \\ (iii) & C_1(\omega,\zeta)\, \mathbb{1}_{\langle\zeta\rangle \le \langle\omega\rangle^\kappa} & \overset{\langle\omega\rangle\to\infty}{\longrightarrow} 0 \end{array} \tag{123}$$

Assume first that $\sigma \in]-\frac{1}{2}, \frac{1}{2}[$. Lemma 11 gives:

$$\begin{array}{llcl} (a) & |\zeta p_0| & \le & C\langle\omega\rangle^2 M_\sigma\left[\|F\|_{\sigma,\zeta} + \|Y\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \\ (b) & \frac{M_{-\sigma-2}}{M_{-\sigma}}\|Y_1\|_{\sigma,\zeta} + \|Y_1\|_{\sigma+2,\zeta} & \le & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\left[\|F\|_{\sigma,\zeta} + \|Y\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \\ (c) & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta} + \|Y_2\|_{\sigma+2,\zeta} & \le & C\left[\|F\|_{\sigma,\zeta} + \|Y\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \\ (d) & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta} + \|\theta\|_{\sigma+2,\zeta} & \le & C\left[\|F\|_{\sigma,\zeta} + |\zeta|\|Y\|_{\sigma,\zeta}\right] \end{array} \tag{124}$$

Let us absorb the perturbative terms of the right-hand side. When $\langle\omega\rangle$ is bounded, estimates (116) are true because $\mathcal{L}$ is an isomorphism. We shall then assume that $\langle\omega\rangle$ is large enough. Absorbing $\|Y_2\|_{\sigma,\zeta}$ in $(124, c)$ is easy and one gets:

$$\|Y_2\|_{\sigma,\zeta} \le C\frac{1}{\langle\omega\rangle^2}\left[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \tag{125}$$

so that (124) becomes:

$$\begin{array}{llcl} (a) & |\zeta p_0| & \le & C\langle\omega\rangle^2 M_\sigma\left[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \\ (b) & \frac{M_{-\sigma-2}}{M_{-\sigma}}\|Y_1\|_{\sigma,\zeta} + \|Y_1\|_{\sigma+2,\zeta} & \le & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\left[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \\ (c) & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta} + \|Y_2\|_{\sigma+2,\zeta} & \le & C\left[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\right] \\ (d) & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta} + \|\theta\|_{\sigma+2,\zeta} & \le & C\left[\|F\|_{\sigma,\zeta} + |\zeta|\|Y_1\|_{\sigma,\zeta} + \frac{|\zeta|^2}{\langle\omega\rangle^2}\|\theta\|_{\sigma,\zeta}\right] \end{array} \tag{126}$$

According to corollary 2, $\langle\omega\rangle^2 M_\sigma M_{-\sigma} \sim \langle\omega\rangle^{-2}$, therefore we can absorb $\|Y_1\|_{\sigma,\zeta}$ in $(126,b)$ to get:

$$\|Y_1\|_{\sigma,\zeta} \leq C\langle\omega\rangle^{-2}\big[\|F\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\big] \tag{127}$$

so that $(126,a,c)$ gives:

$$\begin{array}{lrcl}
(a) & |\zeta p_0| & \leq & C\langle\omega\rangle^2 M_\sigma\big[\|F\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\big] \\
(b) & \frac{M_{-\sigma-2}}{M_{-\sigma}}\|Y_1\|_{\sigma,\zeta} + \|Y_1\|_{\sigma+2,\zeta} & \leq & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big[\|F\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\big] \\
(c) & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta} + \|Y_2\|_{\sigma+2,\zeta} & \leq & C\big[\|F\|_{\sigma,\zeta} + |\zeta|\|\theta\|_{\sigma,\zeta}\big] \\
(d) & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta} + \|\theta\|_{\sigma+2,\zeta} & \leq & C\big[\|F\|_{\sigma,\zeta} + \|\theta\|_{\sigma,\zeta}\big]
\end{array} \tag{128}$$

Absorbing $\|\theta\|_{\sigma,\zeta}$ is then easy and one gets:

$$\|\theta\|_{\sigma,\zeta} \leq C\frac{1}{\langle\omega\rangle^2}\|F\|_{\sigma,\zeta} \tag{129}$$

so that (116) is established and the proof is complete for $\sigma \in]-\frac{1}{2},\frac{1}{2}[$.
Let us now turn to the case $\sigma \in]-\frac{3}{2},-\frac{1}{2}[$. Estimates (120) and lemma 11 with C_1 for Y_1 and C_2 for θ and Y_2 give:

$$\begin{array}{lrcl}
(a) & |\zeta p_0| & \leq & C\langle\omega\rangle^2 M_\sigma\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + \|Y_2\|_{\sigma,\zeta} + |\zeta|C_2 I_\theta\big] \\
(b) & I_1 & \leq & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + \|Y_2\|_{\sigma,\zeta} + |\zeta|C_2 I_\theta\big] \\
(c) & I_2 & \leq & C\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + \|Y_2\|_{\sigma,\zeta} + |\zeta|C_2 I_\theta\big] \\
(d) & I_\theta & \leq & C\big[\|F\|_{\sigma,\zeta} + |\zeta|C_2 I_2 + |\zeta|C_1 I_1\big]
\end{array} \tag{130}$$

where we have denoted

$$\begin{array}{rcl}
I_\theta & = & \langle\omega\rangle^2\|\theta\|_{\sigma,\zeta} + \|\theta\|_{\sigma+2,\zeta} \\
I_2 & = & \langle\omega\rangle^2\|Y_2\|_{\sigma,\zeta} + \|Y_2\|_{\sigma+2,\zeta} \\
I_1 & = & \frac{M_{-\sigma-2}}{M_{-\sigma}}\|Y_1\|_{\sigma,\zeta} + \|Y_1\|_{\sigma+2,\zeta}
\end{array} \tag{131}$$

As previously, absorbing Y_2 in $(130,c)$ gives

$$\|Y_2\|_{\sigma,\zeta} \leq C\frac{1}{\langle\omega\rangle^2}\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|C_2 I_\theta\big] \tag{132}$$

so that we obtain, thanks to $(123,i)$ in (d):

$$\begin{array}{lrcl}
(a) & |\zeta p_0| & \leq & C\langle\omega\rangle^2 M_\sigma\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \\
(b) & I_1 & \leq & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \\
(c) & I_2 & \leq & C\big[\|F\|_{\sigma,\zeta} + \|Y_1\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \\
(d) & I_\theta & \leq & C\big[\,\|F\|_{\sigma,\zeta} + |\zeta|\, C_2\, \|Y_1\|_{\sigma,\zeta} + |\zeta|^2\, C_2^2\, I_\theta + |\zeta|\, C_1\, I_1\,\big]
\end{array} \tag{133}$$

From corollary 2 we have $\langle\omega\rangle^2 M_\sigma M_{-\sigma} \le \langle\omega\rangle^{-1}$ for $\sigma \in]-\frac{3}{2},-\frac{1}{2}[$, thus we can absorb Y_1 in $(133, b)$ to obtain:

$$\|Y_1\|_{\sigma,\zeta} \;\le\; C\langle\omega\rangle^2 M_\sigma M_{-\sigma}\big[\|F\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \tag{134}$$

and

$$\begin{array}{lrcl} (a) & |\zeta p_0| & \le & C\langle\omega\rangle^2 M_\sigma\big[\|F\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \\ (b) & I_1 & \le & C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big[\|F\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \\ (c) & I_2 & \le & C\big[\|F\|_{\sigma,\zeta} + |\zeta|\, C_2\, I_\theta\big] \end{array} \tag{135}$$

Let us now examine (d). Using lemma 11 and corollary 2 we get:

$$\begin{array}{llll} (d) & I_\theta & \le & C\big[\, \|F\|_{\sigma,\zeta} + |\zeta|\, C_2\, \|Y_1\|_{\sigma,\zeta} + |\zeta|^2\, C_2^2\, I_\theta + |\zeta|\, C_1\, I_1\,\big] \\ & & \le & C\,\|F\|_{\sigma,\zeta}\,\big[1 + |\zeta|\, C_2\, \langle\omega\rangle^2 M_\sigma M_{-\sigma} + |\zeta|\, C_1\, \langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big] + \\ & & & C\,|\zeta|\, C_2\, I_\theta\,\big[|\zeta|\, C_2\, \langle\omega\rangle^2\, M_\sigma M_{-\sigma} + |\zeta|\, C_2 + |\zeta|\, C_1\;\, \langle\omega\rangle^2\, M_\sigma M_{-\sigma-2}\,\big] \\ & & \le & C\,\big[1 + |\zeta|\, C_1\, \langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\big]\,\big[\,\|F\|_{\sigma,\zeta} +\, |\zeta|\, C_2\, I_\theta\,\big] \\ & & \le & C\,\big[1 + |\zeta|\, C_1\, \mathbb{1}_{\langle\zeta\rangle \ge \langle\omega\rangle^\kappa} + C_1\, \mathbb{1}_{\langle\zeta\rangle \le \langle\omega\rangle^\kappa}\big]\,\big[\,\|F\|_{\sigma,\zeta} +\, |\zeta|\, C_2\, I_\theta\,\big] \\ & & \le & C\,\big[\,\|F\|_{\sigma,\zeta} +\, |\zeta|\, C_2\, I_\theta\,\big] \end{array} \tag{136}$$

According to corollary 2, $|\zeta|\, C_2$ tends to 0 as $\langle\omega\rangle$ goes to infinity, we thus absorb I_θ in (d) then in (a, b, c) to conclude.
□

Proof of lemma 11. Let $s < \frac{1}{2}$ and $\varphi \in H_\zeta^{s+2}$, with $\varphi = \sum_l \varphi_l e_l(z)$. Define $\phi = \int_0^z \varphi(z')\, dz' = \sum_k \phi_k e_k(z)$. Let us evaluate ϕ_k:

$$\begin{array}{rcl} \phi_k & = & C_0 \int_0^a \phi(z)\sin(\frac{k\pi z}{a})\, dz \\ & = & C_0 \int_0^a \big[\int_0^z \sum_l \varphi_l \sin(\frac{l\pi z'}{a})\, dz'\big]\sin(\frac{k\pi z}{a})\, dz \\ & = & C \int_0^a \big[\sum_l \varphi_l \frac{1}{l}(1-\cos(\frac{l\pi z}{a}))\big]\sin(\frac{k\pi z}{a})\, dz \\ & = & C\sum_l \int_0^a \frac{\varphi_l}{l}\big[\sin(\frac{k\pi z}{a}) - \frac{1}{2}(\sin(\frac{(k+l)\pi z}{a}) + \sin(\frac{(k-l)\pi z}{a}))\big]\, dz \\ & = & C\sum_l \frac{\varphi_l}{l}\big[\frac{1-(-1)^k}{k} - \frac{(1-(-1)^{k+l})k}{k^2-l^2}\big] \end{array} \tag{137}$$

therefore

$$|\phi_k| \sim C\sum_l \frac{|\varphi_l|}{l}\Big[\frac{1}{k} + \frac{k}{(k+l)(|k-l|+1)}\Big] \tag{138}$$

Let us first examine the case $s \in]-\frac{1}{2},\frac{1}{2}[$. We have

$$\begin{array}{rcl} \|\phi\|_s^2 & \sim & \sum_k (1+k^2+\zeta^2)^s \Big(\sum_l \frac{|\varphi_l|}{l}\big[\frac{1}{k} + \frac{k}{(k+l)(|k-l|+1)}\big]\Big)^2 \\ & \le & \|\varphi\|_s^2 \sum_k (1+k^2+\zeta^2)^s \sum_l \frac{1}{l^2(1+l^2+\zeta^2)^s}\big[\frac{1}{k} + \frac{k}{(k+l)(|k-l|+1)}\big]^2 \end{array} \tag{139}$$

When l is much smaller or much larger than k, $[\frac{1}{k}+\frac{k}{(k+l)(|k-l|+1)}]$ is of the same order as $\frac{1}{k}$; when l and k are of the same order, $[\frac{1}{k}+\frac{k}{(k+l)(|k-l|+1)}]$ is of the same order as 1. Therefore we get:

$$\begin{aligned} \|\phi\|_s^2 &\leq C\,\|\varphi\|_s^2 \textstyle\sum_k (1+k^2+\zeta^2)^s \left(\frac{1}{k^2} \sum_l \left(\frac{1}{l^2(1+l^2+\zeta^2)^s}\right) + \frac{1}{k^2(1+k^2+\zeta^2)^s} \right) \\ &\leq C\,\|\varphi\|_s^2 \left(\textstyle\sum_k \frac{(1+k^2+\zeta^2)^s}{k^2} \sum_l \frac{1}{l^2(1+l^2+\zeta^2)^s} + \sum_k \frac{1}{k^2} \right) \end{aligned} \tag{140}$$

For s positive one has:

$$\begin{aligned} \textstyle\sum_k \frac{(1+k^2+\zeta^2)^s}{k^2} &\sim \int_1^\zeta \frac{(x^2+\zeta^2)^s}{x^2}dx + \int_\zeta^{+\infty} \frac{(x^2+\zeta^2)^s}{x^2}dx \\ &\sim \zeta^{2s}+\zeta^{2s-1} \quad \sim \quad \zeta^{2s} \end{aligned} \tag{141}$$

For s positive, we have also:

$$\textstyle\sum_l \frac{1}{l^2(1+l^2+\zeta^2)^s} \quad \sim \quad \zeta^{-2s} \tag{142}$$

so finally one gets, for $s \in]-\frac{1}{2},\frac{1}{2}[$:

$$\textstyle\sum_l \frac{(1+l^2+\zeta^2)^s}{l^2} \quad \sim \quad \zeta^{2s} \tag{143}$$

such that (121) is established, for $s \in]-\frac{1}{2},\frac{1}{2}[$.
We turn now to the case $s \in]-\frac{3}{2},-\frac{1}{2}[$. Let us set

$$\begin{aligned} I_\varphi^2 &= g_{\omega,\zeta}^2 \|\varphi\|_{s,\zeta}^2 + \|\varphi\|_{s+2,\zeta}^2 \\ &= \textstyle\sum_l (1+l^2+\zeta^2)^s (g_{\omega,\zeta}^2 + (1+l^2+\langle\zeta\rangle^2)^2)|\varphi_l|^2 \\ &\sim \textstyle\sum_l (1+l^2+\zeta^2)^s ((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2 + l^4)|\varphi_l|^2 \end{aligned} \tag{144}$$

where $g_{\omega,\zeta}$ can be either $\langle\omega\rangle^2$ or $\frac{M_{-s-2}}{M_{-s}}$.
Let us estimate $\|\phi\|_s$ as a function of I_φ. From (138) and the Cauchy-Schwarz inequality one has:

$$\|\phi\|_s^2 \leq I_\varphi^2 \textstyle\sum_k (1+k^2+\zeta^2)^s \sum_l \frac{1}{l^2(1+l^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+l^4)} \left[\frac{1}{k}+\frac{k}{(k+l)(|k-l|+1)}\right]^2 \tag{145}$$

As previously and as for the estimation of M_σ, we have

$$\begin{aligned} &\sum_{l<<k \text{ or } l>>k} \frac{1}{l^2(1+l^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+l^4)} \left[\frac{1}{k}+\frac{k}{(k+l)(|k-l|+1)}\right]^2 \\ &\qquad \leq \quad C \quad \frac{1}{k^2} \sum_l \frac{1}{l^2(1+l^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+l^4)} \\ &\qquad \leq \quad C \quad \frac{1}{k^2}[I_1+I_2+I_3] \end{aligned} \tag{146}$$

with

$$
\begin{aligned}
I_1 &= \int_1^\zeta \frac{dx}{x^2(1+x^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+x^4)} \sim \frac{\zeta^{-2s}}{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^2} \\
I_2 &= \int_\zeta^{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{1/2}} \frac{dx}{x^2(1+x^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+x^4)} \\
&\sim \frac{1}{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^2}\int_\zeta^{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{1/2}} \frac{dx}{x^{2s+2}} \sim \frac{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{-s-1/2}}{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^2} \\
I_3 &= \int_{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{1/2}}^{+\infty} \frac{dx}{x^2(1+x^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+x^4)} \sim \frac{1}{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{s+5/2}}
\end{aligned}
\tag{147}
$$

Hence

$$
\begin{aligned}
\sum_{l<<k \text{ or } l>>k} \frac{1}{l^2(1+l^2+\zeta^2)^s(g_{\omega,\zeta}^2+l^4+\zeta^4)}\left[\frac{1}{k}+\frac{k}{(k+l)(|k-l|+1)}\right]^2 \\
\leq \quad C \quad \frac{1}{k^2}\frac{\zeta^{-2s}+(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{-s-1/2}}{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^2}
\end{aligned}
\tag{148}
$$

We have also:

$$
\sum_k (1+k^2+\zeta^2)^s \frac{1}{k^2} \sim \zeta^{2s} \tag{149}
$$

Let us now examine (145) for l and k of the same order:

$$
\begin{aligned}
&\textstyle\sum_k (1+k^2+\zeta^2)^s \displaystyle\sum_{k,l\simeq k} \frac{1}{l^2(1+l^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+l^4)}\left[\frac{1}{k}+\frac{k}{(k+l)(|k-l|+1)}\right]^2 \\
&\sim \sum_k (1+k^2+\zeta^2)^s \frac{1}{k^2(1+k^2+\zeta^2)^s((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+k^4)}\left[\frac{1}{k}+C\right]^2 \\
&\sim \sum_k \frac{1}{k^2((g_{\omega,\zeta}+\langle\zeta\rangle^2)^2+k^4)} \\
&\leq C\frac{1}{(g_{\omega,\zeta}+\langle\zeta\rangle^2)^2}
\end{aligned}
\tag{150}
$$

We have proved that, for $s\in]-\frac{3}{2},-\frac{1}{2}[$:

$$
\|\phi\|_s \quad \leq \quad I_\varphi \quad \left((g_{\omega,\zeta}+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^s(g_{\omega,\zeta}+\langle\zeta\rangle^2)^{-s/2-5/4}\right) \tag{151}
$$

Let us set

$$\begin{array}{rcl} C_2(\omega,\zeta) & = & (\langle\omega\rangle^2+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^s(\langle\omega\rangle^2+\langle\zeta\rangle^2)^{-s/2-5/4} \\ C_1(\omega,\zeta) & = & (\frac{M_{-s-2}}{M_{-s}}+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^s(\frac{M_{-s-2}}{M_{-s}}+\langle\zeta\rangle^2)^{-s/2-5/4} \end{array} \tag{152}$$

We have easily, considering separately the cases $s\geq -1$ and $s\leq -1$:

$$\begin{array}{rcl} |\zeta|\, C_2 & \sim & |\zeta|\,(\langle\omega\rangle^2+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^{s+1}(\langle\omega\rangle^2+\langle\zeta\rangle^2)^{-s/2-5/4} \\ & \leq & C\langle\omega\rangle^{-1}+\langle\zeta\rangle^{s+1}\langle\omega\rangle^{-s-5/2} \\ & \leq & C\langle\omega\rangle^{-1} \end{array} \tag{153}$$

which gives in particular $(123,i)$.
Let us now turn to C_1, and set

$$\kappa\;=\;\frac{2(-s-2)+1}{2(-s-2)}\;=\;\frac{2s+3}{2s+4} \tag{154}$$

For $s\in]-\frac{3}{2},-\frac{1}{2}[$ we have:

$$\begin{array}{rcl} \frac{M_{-s-2}}{M_{-s}} & \sim & (\langle\zeta\rangle^{s+2}\langle\omega\rangle^{-2}\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\omega\rangle^{s-1/2}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa})\langle\zeta\rangle^{-s}\langle\omega\rangle^2 \\ & \sim & \langle\zeta\rangle^2\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}+\langle\zeta\rangle^{-s}\langle\omega\rangle^{s+3/2}\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa} \end{array} \tag{155}$$

Thanks to (152), we get:

$$\langle\zeta\rangle\, C_1\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa}\;\;\sim\;\;(\langle\zeta\rangle^{-2}+\langle\zeta\rangle^{-5/2})\,\langle\zeta\rangle\,\mathbb{1}_{\langle\zeta\rangle\geq\langle\omega\rangle^\kappa} \tag{156}$$

which gives $(123,ii)$, as $\kappa>0$. Then

$$\begin{array}{rcl} C_1\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa} & \sim & (\langle\zeta\rangle^{-s}\langle\omega\rangle^{s+3/2}+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^s(\langle\zeta\rangle^{-s}\langle\omega\rangle^{s+3/2}+\langle\zeta\rangle^2)^{-s/2-5/4} \\ & \leq & (\langle\omega\rangle^{s+3/2}+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^s(\langle\zeta\rangle^{-s}\langle\omega\rangle^{s+3/2}+\langle\zeta\rangle^{-s})^{-s/2-5/4} \\ & \leq & (\langle\omega\rangle^{s+3/2}+\langle\zeta\rangle^2)^{-1}+\langle\zeta\rangle^{s^2/2+9s/4}(\langle\omega\rangle^{s+3/2}+1)^{-s/2-5/4} \end{array} \tag{157}$$

with $\langle\zeta\rangle^{-s}>1$ and $\langle\zeta\rangle^2>\langle\zeta\rangle^{-s}$. Finally we notice that

$$s+\tfrac{3}{2}>0\quad;\quad\tfrac{1}{2}s^2+\tfrac{9}{4}s<-1\quad;\quad-\tfrac{1}{2}s-\tfrac{5}{4}<-\tfrac{1}{2} \tag{158}$$

therefore $C_1\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}$ tends to 0 as $\langle\omega\rangle$ tends to infinity and the lemma is proven.
□

3.5 End of the proof

We are now in a position to complete the proof, thanks to estimates (116) of thm 3.
For u and v one gets:

$$\begin{aligned} \|Y_1\|_{\sigma+2,\zeta} &\leq C\langle\omega\rangle^2 M_\sigma M_{-\sigma-2}\|F\|_{\sigma,\zeta} \\ \|Y_2\|_{\sigma+2,\zeta} &\leq C\|F\|_{\sigma,\zeta} \end{aligned} \tag{159}$$

According to corollary 2, $\langle\omega\rangle^2 M_\sigma M_{-\sigma-2} \leq C$, therefore

$$\|Y\|_{\sigma+2,\zeta} \leq C\|F\|_{\sigma,\zeta} \tag{160}$$

Inverting the Fourier-Laplace transform, one gets

$$\begin{aligned} &Y \in L^2(\mathbb{R}_+; H_\zeta^{\sigma+2}) \\ \|Y\|_{L^2(\mathbb{R}_+;H_\zeta^{\sigma+2})} &\leq C\|F\|_{L^2(\mathbb{R}_+;H_\zeta^\sigma)} \end{aligned} \tag{161}$$

hence (57) follows, for u and v.
The same argument applies to θ:

$$\begin{aligned} &\theta \in L^2(\mathbb{R}_+; H_\zeta^{\sigma+2}) \\ \|\theta\|_{L^2(\mathbb{R}_+;H_\zeta^{\sigma+2})} &\leq C\|F\|_{L^2(\mathbb{R}_+;H_\zeta^\sigma)} \end{aligned} \tag{162}$$

and (57) is established for θ, so that (16) and (17) are proven.
For $\sigma \in]-\frac{1}{2}, \frac{1}{2}[$, one has $M_\sigma \sim \frac{\langle\zeta\rangle^{-\sigma}}{\langle\omega\rangle^2}$ and with (116,a) one obtains

$$\langle\zeta\rangle^{\sigma+1}|p_0| \leq C\|F\|_{\sigma,\zeta} \tag{163}$$

then $p_0 = q$ satisfies

$$\begin{aligned} &q \in L^2(\mathbb{R}_+; H^{\sigma+1}(\mathbb{T}^2)) \\ \|q\|_{L^2(\mathbb{R}_+;H^{\sigma+1}(\mathbb{T}^2))} &\leq C\|F\|_{L^2(\mathbb{R}_+;\mathcal{H}^\sigma)} \end{aligned} \tag{164}$$

hence (58) and (20) follow, for $\sigma \in]-\frac{1}{2}, \frac{1}{2}[$.
For $\sigma \in]-\frac{3}{2}, -\frac{1}{2}[$, we split $p_0 = q$ in $q_1 + q_2$ as follows:

$$q_1 = p_0 \, \mathbb{1}_{\langle\zeta\rangle \leq \langle\omega\rangle^\kappa}, \quad q_2 = p_0 \, \mathbb{1}_{\langle\zeta\rangle \geq \langle\omega\rangle^\kappa} \tag{165}$$

Similarly, lemma 9 gives for q_2:

$$\langle\zeta\rangle^{\sigma+1}|q_2| \leq C\|F\|_{\sigma,\zeta} \tag{166}$$

thus

$$\begin{aligned} q_2 &\in L^2(\mathbb{R}_+; H^{\sigma+1}(\mathbb{T}^2)) \\ \|q_2\|_{L^2(\mathbb{R}_+;H^{\sigma+1}(\mathbb{T}^2))} &\leq C\|F\|_{L^2(\mathbb{R}_+;\mathcal{H}^\sigma)} \end{aligned} \tag{167}$$

For q_1, lemma 9 gives:

$$\langle\omega\rangle^{1/2+\sigma}\langle\zeta\rangle|q_1| \leq C\|F\|_{\sigma,\zeta} \tag{168}$$

As

$$\kappa = \frac{2\sigma+1}{2\sigma} \in \,]0, \frac{2}{3}[\tag{169}$$

we get

$$\langle\zeta\rangle \leq \langle\omega\rangle^\kappa = (|\tau| + \langle\zeta\rangle^2)^{1/2} \Rightarrow \langle\zeta\rangle \leq |\tau|^{1/2} \tag{170}$$

Therefore $\langle\omega\rangle \sim \langle\tau\rangle$ in the support of q_1, so (168) becomes

$$\begin{aligned} q_1 &\in H^{\sigma/2+1/4}(\mathbb{R}_+; H^1(\mathbb{T}^2)) \\ \|q_1\|_{H^{\sigma/2+1/4}(\mathbb{R}_+;H^1(\mathbb{T}^2))} &\leq C\|F\|_{L^2(\mathbb{R}_+;\mathcal{H}^\sigma)} \end{aligned} \tag{171}$$

□

4 Remarks and further results

4.1 An explicit formula for the pressure

Study of the uncoupled system gives the following formula for p_0:

$$p_0 = -\frac{1}{a}\frac{\omega^2}{\zeta^2}\Big[\mathcal{N}(\frac{\omega a}{\sqrt{\nu}})\Big]^{-1}\int_0^a (\omega^2 - \nu\partial_{zz})^{-1}[i\xi f_1 + i\eta f_2]\,dz \tag{172}$$

where $\mathcal{N}$ is defined by formula (110). For the coupled system, (119) enables us to write

$$\begin{aligned} p_0 = -\tfrac{1}{a}\tfrac{\omega^2}{\zeta^2}\big[\mathcal{N}(\tfrac{\omega a}{\sqrt{\nu}})\big]^{-1}\textstyle\int_0^a(\omega^2 - \nu\partial_{zz})^{-1}[i\xi(f_1 + \alpha v - i\xi\beta\int_0^z\theta) \\ +i\eta(f_2 - \alpha u - i\eta\beta\textstyle\int_0^z\theta)]\,dz \end{aligned} \tag{173}$$

According to theorem 3, the terms involving u and θ are smooth enough that the singular term of the pressure is given by

$$q_1(\tau,\xi,\eta) = -\,\mathbb{1}_{\langle\zeta\rangle\leq\langle\omega\rangle^\kappa}\,\frac{1}{a}\frac{\omega^2}{\zeta^2}\Big[\mathcal{N}(\frac{\omega a}{\sqrt{\nu}})\Big]^{-1}\int_0^a(\omega^2 - \nu\partial_{zz})^{-1}[i\xi f_1 + i\eta f_2]\,dz \tag{174}$$

with $\mathcal{N}$ defined by (110).

4.2 Proof of corollary 1

Under the assumptions of corollary 1, φX satisfies equation (15) with right-hand side $\varphi F + \varphi'(t)X$ in $(L^2(0,T;\mathcal{H}^{-1}))^3$ and with support in $t > 0$. Theorem 1 gives (25). Thanks to the previous remark, we know that q splits in $q_1 + q_2$, with q_1 less smooth than q_2, and we have explicitly (in Fourier variables)

$$q_1(\tau,\xi,\eta) = -\,\mathbb{1}_{\langle\zeta\rangle\le\langle\omega\rangle^\kappa}\,\frac{1}{a}\frac{\omega^2}{\zeta^2}\Big[\mathcal{N}(\frac{\omega a}{\sqrt{\nu}})\Big]^{-1}\int_0^a(\omega^2-\nu\partial_{zz})^{-1}[i\xi(\varphi f_1)_{\tau,\zeta}+i\eta(\varphi f_2)_{\tau,\zeta}]\,dz \tag{175}$$

(indeed, the contribution of $\varphi'(t)X$ is smooth, because $C \in L^2(0,T;\mathcal{V})$)
Thus the following equivalence holds:

$$\begin{array}{rl} & q_1 \in L^2(0,T;L^2(\mathbb{T}^2)) \\ \Leftrightarrow & \frac{1}{\zeta^2}\int_0^a(\omega^2-\nu\partial_{zz})^{-1}[i\xi(\varphi f_1)_{\tau,\zeta}+i\eta(\varphi f_2)_{\tau,\zeta}]\,dz \in L^2(\tau;\ell^2_\zeta) \\ \Leftrightarrow & \Delta_2^{-1}\big[\int_0^a(\partial_t-\nu\Delta)^{-1}[\varphi\partial_x f_1+\varphi\partial_y f_2]\,dz\big] \in L^2(0,T;L^2(\mathbb{T}^2)) \end{array} \tag{176}$$

□

4.3 A counter-example to maximal estimates

In this paragraph, we construct a counter-example to the maximal estimate (23) for $\sigma < -1/2$. It is sufficient to find $F \in (L^2(\mathbb{R},\mathcal{H}^\sigma))^3$ such that the gradient of the associated p_0 is not in L^2 in time, as each other term of the equations (15)(except $\partial_t X$) are in L^2 in time.
Let $\sigma \in]-\frac{3}{2},-\frac{1}{2}[$ and let α be such that $\alpha \in]\sigma+\frac{1}{2},0[$. Let $g(t) \in L^2(\mathbb{R})$ with support in $t > 0$. Let $f \in \mathcal{H}^\sigma$, independent of time, be defined by

$$f(x,y,z) = \sum_k k^{-\alpha}\,e_k(z)e^{ix} \tag{177}$$

If $\zeta = (1,0)$ then $f_{\zeta,k} = k^{-\alpha}$ and if $\zeta \ne (1,0)$ then $f_{\zeta,k} = 0$.
Now let $F \in (L^2(\mathbb{R},\mathcal{H}^\sigma))^3$ with support in $t > 0$ defined by $F = (fg,0,0)$ and let p be the pressure, solution of (15) with right hand side F. We write $p = q_1 + p_1$ where ∇p_1 is as smooth as F is, and q_1 is explicitly given by formula (174) (omitting the high frequency cut-off):

$$q_1(\tau,\xi,\eta) = -\frac{1}{a}\frac{\omega^2}{\zeta^2}\Big[\mathcal{N}(\frac{\omega a}{\sqrt{\nu}})\Big]^{-1}\int_0^a(\omega^2-\nu\partial_{zz})^{-1}[i\xi f(\xi,\eta,z)g(\tau)]\,dz \tag{178}$$

Therefore $q_1(t,x,y) = q_1(t)\,e^{ix}$ and the Fourier transform of $q_1(t)$ is

$$\begin{array}{rcl} q_1(\tau) & = & g(\tau)\,m(\tau) \\ \textit{with}\;\; m(\tau) & = & \frac{1}{a}(\tau-i)\Big[\mathcal{N}(\frac{\omega a}{\sqrt{\nu}})\Big]^{-1}\int_0^a(\omega^2-\nu\partial_{zz})^{-1}[f_{(1,0)}(z)]\,dz \end{array} \tag{179}$$

with $\omega^2 = i\tau + \zeta^2$. Let us now calculate $m(\tau)$:

$$\begin{aligned} m(\tau) &= \tfrac{1}{a}(\tau - i)\Big[\mathcal{N}(\tfrac{\omega a}{\sqrt{\nu}})\Big]^{-1} \int_0^a \sum_k \frac{f_k}{\omega^2 + \frac{\nu k^2\pi^2}{a^2}} e_k(z)\, dz \\ &= \tfrac{C_a}{a}(\tau - i)\Big[\mathcal{N}(\tfrac{\omega a}{\sqrt{\nu}})\Big]^{-1} \sum_{k \text{ odd}} \frac{k^{-\alpha}}{k\,(1 + i\tau + \frac{\nu k^2\pi^2}{a^2})} \end{aligned} \tag{180}$$

In order to get a lower bound on $|m(\tau)|$, let us define

$$\begin{aligned} S_k &= \sum_{k \text{ odd}} \frac{k^{-\alpha}}{k\,(1 + i\tau + \frac{\nu k^2\pi^2}{a^2})} \\ &= \sum_{k \text{ odd}} \frac{k^{-\alpha}(1 + \frac{\nu k^2\pi^2}{a^2})}{k\,((1 + \frac{\nu k^2\pi^2}{a^2})^2 + \tau^2)} - i \sum_{k \text{ odd}} \frac{k^{-\alpha}\tau}{k\,((1 + \frac{\nu k^2\pi^2}{a^2})^2 + \tau^2)} \end{aligned} \tag{181}$$

We have, as $\alpha + 3 > 1$:

$$\begin{aligned} |(\tau - i)S_k| &\geq |\Re((\tau - i)S_k)| = |\tau\Re(S_k) + \Im(S_k)| \\ &= |\tau| \sum_{k \text{ odd}} \frac{k^{-\alpha}}{k\,((1 + \frac{\nu k^2\pi^2}{a^2})^2 + \tau^2)}(1 + \frac{\nu k^2\pi^2}{a^2} - 1) \\ &= C|\tau| \sum_{k \text{ odd}} \frac{k^{2-\alpha}}{k\,((1 + \frac{\nu k^2\pi^2}{a^2})^2 + \tau^2)} \\ &\geq C|\tau| \sum_{k \geq |\tau|^{1/2}} \frac{k^{2-\alpha}}{k\,((1 + \frac{\nu k^2\pi^2}{a^2})^2 + \tau^2)} \\ &\sim |\tau| \int_{|\tau|^{1/2}} \frac{x^{2-\alpha}\, dx}{x(x^4 + \tau^2)} \\ &\sim |\tau|^{-\alpha/2} \end{aligned} \tag{182}$$

Moreover we know that $\mathcal{N}^{-1} \to 1$ as $|\tau| \to +\infty$, thus we have, if $|\tau|$ is large enough

$$|m(\tau)| \geq C|\tau|^{-\alpha/2} \tag{183}$$

So finally, for $|\tau|$ large enough

$$|q_1(\tau)| \;\geq\; C\,|g(\tau)|\,|\tau|^{-\alpha/2} \tag{184}$$

Choose now $g \in L^2(\mathbb{R})$, with support in $t > 0$, such that $g \notin H^s(\mathbb{R})$ for all $s > 0$. Then $|g(\tau)|\,|\tau|^{-\alpha/2}$ is not in $L^2(\tau \in \mathbb{R})$ (because $-\alpha > 0$), and the pressure $q_1(t)e^{ix}$ is not L^2 in time and neither is its gradient.

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